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A Unified View of Consequence Relation, Belief Revision and Conditional Logic

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A Unified View of Consequence Relation, Belief Revision and Conditional Logic*

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Abstract

The notion of *minimality* is widely used in three different areas of Artificial Intelligence: nonmonotonic reasoning, belief revision, and conditional reasoning. However, it is difficult for the readers of the literature in these areas to perceive the similarities clearly, because each formalization in those areas uses its own language sometimes without referring to other formalizations. We define *ordered structures* and *families of ordered structures* as the common ingredient of the semantics of all the works above. We also define the logics for ordered structures and families. We present a uniform view of how minimality is used in these three areas, and shed light on deep reciprocal relations among different approaches of the areas by using the ordered structures and the families of ordered structures.

1 Introduction

The notion of *minimality* is proving to be a key unifying idea in three different areas of Artificial Intelligence: nonmonotonic reasoning, belief revision, and conditional reasoning. However, it is difficult for the readers of the literature in these areas to perceive the similarities clearly. The models used differ, sometimes superficially, sometimes in depth, and the notation is different, making it hard to apply results on, say, conditional logic to, say, belief revision. Even within the same area there is confusion, as, for example, different authors use different formalisms for conditional logic, sometimes without relating their proposals to the literature. We present a uniform view of how minimality is used in these three areas, shedding light on deep connections among the areas. We clarify differences and similarities between different approaches by classifying them according to the notion of minimality that they are based on.

The first field in which minimality plays a crucial role is nonmonotonic reasoning. Shoham [1987] proposes a uniform approach to subsuming various formalisms of nonmonotonic reasoning in terms of *preferential relations* among interpretations. Kraus, Lehmann and Magidor [1990] propose several *consequence relations* that capture general patterns of nonmonotonic reasoning. A consequence relation, denoted by $\mu \mathrel{|\!\sim} \phi$, means that μ is a good enough reason to believe ϕ, or that ϕ is a plausible consequence of μ. We can regard their work as an extension of Shoham's work since some consequence relations can be characterized in terms of preferential relations among possible worlds.

The second field in which minimality is discussed is knowledge base *revision* and *update*. Alchourrón, Gärdenfors and Makinson [1985] propose, on philosophical grounds, a set of rationality postulates that belief revision operators must satisfy. Katsuno and Mendelzon [1989; 1991a] show that the AGM postulates precisely characterize revision operators that accomplish a modification with minimal change among models of knowledge bases expressed in a finitary propositional logic.

Katsuno and Mendelzon [1991b] clarify a fundamental distinction between knowledge base revision and update. They propose update postulates in the spirit of the AGM revision postulates, and show that the update postulates precisely characterize minimal change update operators just as the revision postulates characterize minimal change revision.

The third field is conditional logic, which is concerned with the logical and semantical properties of *counterfactuals*, statements such as "if I were a bird then I could fly." Many applications of counterfactuals in Artificial Intelligence are pointed out by Ginsberg [1986]. Delgrande [1988] uses a conditional logic to formalize default reasoning.

There are several different conditional logics, and some of them can be formulated in terms of minimal change [Nute, 1984]. According to this minimality view, a counterfactual, $\mu > \phi$, is true if we add its antecedent μ to our set of beliefs, modify the set as little as possible to preserve consistency, and then its consequent ϕ is true under the modified set of beliefs. Gärdenfors [1988] shows the difficulties associated with using revision to model this view of counterfactuals; Katsuno and Mendelzon [1991b] suggest using update instead of revision. This suggestion is carried out by Grahne [1991] in a logic of updates and counterfactuals.

Lewis [1973] proposes conditional logics called VC and VW. In the semantics of VC and VW, total pre-orders (in Lewis' terminology, "systems of spheres") play a key role. Pollock [Pollock, 1981; Nute, 1984] proposes another conditional logic, called SS, the semantics of which is determined by changing total pre-orders in VC to partial orders.

In this paper, we discuss fundamental similarities and differences among the above works, and give a unified view of all the above in terms of notions of minimality.

A good deal of previous work has been done, especially recently, on comparisons among consequence relations, knowledge base revision and conditional logics. Kraus, Lehmann and Magidor [1990] point out a relationship between one of their consequence relations and conditional logics. We extend their result and show more accurate semantic and syntactic correspondences.

Makinson and Gärdenfors [Makinson and Gärdenfors, 1991; Gärdenfors, 1990] show the relationship between the AGM postulates for revision and postulates for non-monotonic consequence relations. Gärdenfors [1988] also investigates the relationship between knowledge base revision and conditional logic based on Ramsey test in terms of syntax. He shows that although there are close resemblances between his postulates for revision and axioms of conditional logic, it is impossible to formulate conditional logic by knowledge base revision and Ramsey test. We show the distinctions between conditional logic and knowledge base revision in light of their semantics, and show that there is a close relationship between knowledge base update and conditional logic. Grahne [1991] proposes a new logic in which counterfactual and update are treated in a unified way.

Bell [1989] proposes using a conditional logic, called C, to extend Shoham's work. The C logic is no more than SS, although the semantics of C seems to be slightly different from the semantics of SS.

Boutilier [1990a; 1990b] establishes a mapping between two conditional logics, called CT4 and CT4D, and consequence relations. We show the differences between his conditional logics and other conditional logics in light of notions of minimality. We also extend these results to other conditional logics.

The outline of the paper is as follows. We define, in Section 3, *ordered structures* and *families of ordered structures* as the common ingredient of the semantics of all the works above, based on the observation that an order among possible worlds is a key concept in all of them. We define logics for ordered structures and families, and give sound and complete axiomatizations for them. In Section 4, we consider in detail each of the three areas - nonmonotonic logic, belief revision, and conditional logic – and show how the works in each area fit into our framework. Finally, in Section 5 we use our framework to derive new cross-connections among the areas.

2 Preliminaries

Let L be a language of propositional logic that may have an infinite number of propositional letters. Let L_n be a finitary propositional language that has exactly n propositional letters. We denote the set consisting of all the interpretations of L_n by $\mathcal{I}_n$. Throughout this paper, we use ψ, μ and ϕ to denote a well-formed formula of L.

A pre-order $\leq$ over a set W is a reflexive and transitive relation. We denote the strict order of $\leq$ by $<$. An element w is a *minimum* with respect to $\leq$ if $w < w'$ for any $w' \in W - \{w\}$. Let W' be a subset of W. An element

w is *minimal* in W' with respect to $\leq$ if w is a member of W' and for any $w' \in W'$, $w' \leq w$ implies $w \leq w'$. We denote by $Min(W', \leq)$ the set of minimal elements in W' with respect to $\leq$. We can define a total pre-order and a partial order in a usual way.

3 Common Language

We define a common language $L_{\leadsto}$ to discuss consequence relations, knowledge base revision, update and conditional logics in a unified way. The language $L_{\leadsto}$ is a language augmented L with a binary connective $\leadsto$. The connective $\leadsto$ is directly related to the connective $\mid\!\sim$ in the context of consequence relations and the conditional connective $>$ of conditional logics. We also use the notation $L_{n,\leadsto}$ to denote the language augmented L_n with $\leadsto$.

We define $Wff_{\leadsto}$ as the set consisting of all the well formed formulas of $L_{\leadsto}$. We also define $S\text{-}Wff_{\leadsto}$ as the set of formulas that have no nesting of $\leadsto$. For example, $\phi \leadsto (\psi \leadsto \mu)$ is not a formula of $S\text{-}Wff_{\leadsto}$. We define $Wff_{n,\leadsto}$ as the set consisting of all the formulas in $Wff_{\leadsto}$ such that every propositional letter occurred in the formula is a propositional letter of L_n. Let $Wff^m_{n,\leadsto}$ be the set of all the formulas in $Wff_{\leadsto}$ such that the depth of $\leadsto$ in the formula is at most m.

3.1 Ordered Structure

An ordered structure is one of the central notions in the semantics of the works that involve minimality. We can give semantics of the formulas in $S\text{-}Wff_{\leadsto}$ by using ordered structures. An ordered structure is a special case of Kripke structure used in modal logics.

We define an *ordered structure* $\mathcal{O}$ as a triple $\langle W, \leq, V \rangle$, where W is a nonempty set of worlds, $\leq$ is a pre-order over W satisfying the smoothness property defined later, and V is a function that maps a pair of each propositional letter of L and an element of W to $\mathtt{T}$ or $\mathtt{F}$.

For an ordered structure $\mathcal{O} = \langle W, \leq, V \rangle$, we define truth of each formula in $S\text{-}Wff_{\leadsto}$ as follows. First, we recursively define the truth of formulas at a world w by using $\mathcal{O}, w \models$.

$$\begin{array}{lll} \mathcal{O}, w \models p & \text{iff} & V(p, w) = \mathtt{T} \\ \mathcal{O}, w \models \neg\alpha & \text{iff} & \mathcal{O}, w \not\models \alpha, \\ \mathcal{O}, w \models \alpha \wedge \beta & \text{iff} & \mathcal{O}, w \models \alpha \text{ and } \mathcal{O}, w \models \beta, \\ \mathcal{O}, w \models \phi \rightsquigarrow \varphi & \text{iff} & Min(\|\phi\|, \leq) \subseteq \|\varphi\|, \end{array}$$

where $\|\phi\| = \{w \mid V(\phi, w) = \mathtt{T}\}$.

We say that a formula α in $S\text{-}W\!f\!f_{\rightsquigarrow}$ is true under an ordered structure $\mathcal{O}$, denoted by $\mathcal{O} \models \alpha$, if $\mathcal{O}, w \models \alpha$ for any w in $Min(W, \leq)$, that is, α is true at every minimal world of W.

Note the following facts.

1. If $w_1 \leq w_2$ means w_1 is a more natural world than w_2, then the semantics defined above shows that $\phi \rightsquigarrow \varphi$ is true under $\mathcal{O}$ if and only if all most natural worlds that satisfy ϕ also satisfy φ.

2. If W is an infinite set, there might exist infinite descending chains of elements of W. Then, although some $\|\phi\|$ is not empty, $Min(\|\phi\|, \leq)$ might be empty.

The smoothness condition, mentioned in the definition of ordered structure, precludes the possibility of the above emptiness problem. A pre-order $\leq$ satisfies the *smoothness condition* if, for any formula ϕ of L and any $w \in \|\phi\|$, there is some world w' such that $w' \leq w$ and w' is minimal in $\|\phi\|$ with respect to $\leq$.

We consider various restrictions on an ordered structure $\mathcal{O} = \langle W, \leq, V\rangle$. If the pre-order $\leq$ is a total pre-order (or a partial order), then we say that $\mathcal{O}$ is a *totally ordered structure* (or *partially ordered structure*). If W is finite, we say that $\mathcal{O}$ is a *finite ordered structure*. If, for any two different worlds, w_1 and w_2, there is some formula ϕ of L such that $V(\phi, w_1) \neq V(\phi, w_2)$, then we say that $\mathcal{O}$ is a *distinguishable ordered structure*. If $\mathcal{O}$ is a distinguishable ordered structure then no two worlds represent the same interpretation of L. If the pre-order $\leq$ has a minimum in W, we say that $\mathcal{O}$ is an *ordered structure with a minimum*. The ordered structures with a minimum are used to give semantics to formulas of $W\!f\!f_{\rightsquigarrow}$.

The readers who are familiar with ranked models and preferential models of consequence relations may easily notice that totally (resp. partially) ordered structures are very similar to ranked (resp. preferential) models.

Another restriction is the case where a finitary propositional logic L_n is used instead of L. An *ordered L_n-structure* is an ordered structure $\langle W, \leq, V \rangle$ such that if a propositional letter p is not in L_n then $V(p, w)$ is undefined. We can also define various restrictions (total, partial, finite, distinguishable, with a minimum) on ordered L_n-structures.

Next, we consider a collection of ordered structures in order to give semantics to any formula of $Wff_{\leadsto}$. A *family of ordered structures* $\hat{\mathcal{O}} = (\mathcal{O}_w)_{w \in W}$ is a collection of ordered structures such that W is a nonempty set of worlds, each $\mathcal{O}_w = \langle W_w, \leq_w, V \rangle$ is an ordered structure, W_w is a nonempty subset of W, w is minimal in W_w with respect to $\leq_w$, and a stronger smoothness condition defined later is satisfied. For a family of ordered structures $\hat{\mathcal{O}} = (\langle W_w, \leq_w, V \rangle)_{w \in W}$, if each w is the minimum of W_w with respect to $\leq_w$ then we say that $\hat{\mathcal{O}}$ is a *family of ordered structures with minimum.* We can also define various restrictions on a family of ordered structures (with minimum) in a similar way to the case of ordered structures.

For a family of ordered structures $\hat{\mathcal{O}} = (\mathcal{O}_w)_{w \in W}$, we recursively define the truth of formulas at each world w in W as follows.

$$
\begin{array}{lll}
\hat{\mathcal{O}}, w \models p & \text{iff} & V(p, w) = \mathtt{T}, \\
\hat{\mathcal{O}}, w \models \neg A & \text{iff} & \hat{\mathcal{O}}, w \not\models A, \\
\hat{\mathcal{O}}, w \models A \wedge B & \text{iff} & \hat{\mathcal{O}}, w \models A \text{ and } \hat{\mathcal{O}}, w \models B, \\
\hat{\mathcal{O}}, w \models A \leadsto B & \text{iff} & Min(\|A\|^{\hat{\mathcal{O}}} \cap W_w, \leq_w) \subseteq \|B\|^{\hat{\mathcal{O}}},
\end{array}
$$

where $\|A\|^{\hat{\mathcal{O}}} = \{w \mid \hat{\mathcal{O}}, w \models A\}$. Intuitively, the set $\|A\|^{\hat{\mathcal{O}}}$ denotes all the worlds under which A is true. We say that A is true under a family of ordered structures $\hat{\mathcal{O}}$ if $\|A\|^{\hat{\mathcal{O}}} = W$.

The stronger smoothness condition for a family of ordered structures $\hat{\mathcal{O}}$ is: for any formula A in $Wff_{\leadsto}$ and any $w \in W$, if $w' \in \|A\|^{\hat{\mathcal{O}}} \cap W_w$ then there is some world $w'' \in W_w$ such that $w'' \leq w'$ and w'' is minimal in $\|A\|^{\hat{\mathcal{O}}} \cap W_w$ with respect to $\leq_w$.

3.2 Validity

We investigate the relationship between ordered structures and families of ordered structures (with minimum) in light of the validity of formulas. We also show how various restrictions on those structures are related to the interpretation of formulas.

First, we can show that there is no difference between ordered structures and families of ordered structure in light of the validity of formulas of $S\text{-}Wff_{\leadsto}$.

Theorem 3.1 *For any formula α in $S\text{-}Wff_{\leadsto}$, α is valid under totally (resp. partially) ordered structures if and only if α is valid under families of totally (resp. partially) ordered structures.*

The validity under families of ordered structures and the validity under families of ordered structures with minimum are *different* as shown in the following example. We show, at the end of this subsection, that the two validities are equivalent for some restricted formulas.

Example 3.1 Let α be $((p \wedge q) \vee (r \wedge s)) \supset (p \leadsto q) \vee (r \leadsto s)$. Let $\mathcal{O}$ be an ordered structure $\langle W, \leq, V \rangle$ such that $W = \{w_1, w_2\}$, $\leq = \{(w_1, w_1), (w_2, w_2)\}$, and

$$V(w_1, p) = V(w_1, q) = V(w_1, r) = \mathtt{T},\ \ V(w_1, s) = \mathtt{F},$$
$$V(w_2, p) = V(w_2, r) = V(w_2, s) = \mathtt{T},\ \ V(w_2, q) = \mathtt{F}.$$

Then, α is false under $\mathcal{O}$. On the other hand, it is rather easy to show that α is true under any family of ordered structures with minimum.

Next, we can show that all the restrictions on totally ordered structures introduce no distinction as long as we consider formulas of $S\text{-}Wff_{\leadsto}$ in the context of L_n. This result implies that none of the restrictions influence the validity of each formula α.

Theorem 3.2 *For any totally ordered structure $\mathcal{O}$, there exists a finite, distinguishable, totally ordered L_n-structure $\mathcal{O}'$ such that for any formula α in $S\text{-}Wff_{n,\leadsto}$, α is true under $\mathcal{O}$ if and only if α is true under $\mathcal{O}'$.*

The case of partially ordered structures is different from the case of totally ordered structures. The validity under finite, distinguishable, partially ordered L_n-structures is exceptional.

Example 3.2 Let L_2 contain p and q and α be

$$((p \vee q) \leadsto ((\neg p \wedge q) \vee (p \wedge \neg q))) \supset ((p \leadsto \neg q) \vee (q \leadsto \neg p)).$$

Let $\mathcal{O}$ be an ordered structure $\langle W, \leq, V\rangle$ such that $W = \{w_1, w_2, w_3, w_4\}$, $\leq = \{(w_1, w_2), (w_3, w_4)\} \cup \{(w_i, w_i) \mid 1 \leq i \leq 4\}$, and

$$V(w_1, p) = \mathtt{T},\ V(w_1, q) = \mathtt{F},\ V(w_2, p) = V(w_2, q) = \mathtt{T},$$

$$V(w_3, p) = \mathtt{F},\ V(w_3, q) = \mathtt{T},\ V(w_4, p) = V(w_4, q) = \mathtt{T}.$$

Then, α is false under $\mathcal{O}$. On the other hand, we can show that α is true under any finite, distinguishable, partially ordered L_2-structures.

Theorem 3.3 *For any partially ordered structure $\mathcal{O}$, there exists a finite, partially ordered L_n-structure $\mathcal{O}'$ such that for any formula α in S-$Wff_{n,\rightsquigarrow}$, α is true under $\mathcal{O}$ if and only if α is true under $\mathcal{O}'$.*

We can show similar theorems about families of ordered structures (with minimum) in the context of $Wff_{\rightsquigarrow}$.

Theorem 3.4 *For any family of totally ordered structures $\hat{\mathcal{O}}$ (resp. family of totally ordered structures with minimum $\hat{\mathcal{O}}_1$) and any positive integer m, there exists a family of finite, distinguishable, totally ordered L_n-structures $\hat{\mathcal{O}}'$ (resp. family of finite, distinguishable, totally ordered L_n-structures with minimum $\hat{\mathcal{O}}'_1$) such that for any formula A in $Wff^m_{n,\rightsquigarrow}$, A is true under $\hat{\mathcal{O}}$ (resp. $\hat{\mathcal{O}}_1$) if and only if A is true under $\hat{\mathcal{O}}'$ (resp. $\hat{\mathcal{O}}'_1$).*

Theorem 3.5 *For any family of partially ordered structures $\hat{\mathcal{O}}$ (resp. family of partially ordered structures with minimum $\hat{\mathcal{O}}_1$) and any positive integer m, there exists a family of finite partially ordered L_n-structures $\hat{\mathcal{O}}'$ (resp. family of finite partially ordered L_n-structures with minimum $\hat{\mathcal{O}}'_1$) such that for any formula A in $Wff^m_{n,\rightsquigarrow}$, A is true under $\hat{\mathcal{O}}$ (resp. $\hat{\mathcal{O}}_1$) if and only if A is true under $\hat{\mathcal{O}}'$ (resp. $\hat{\mathcal{O}}'_1$).*

Third, we can define a conditional Horn formula by regarding $\mu \rightsquigarrow \phi$ as an atom. For example, a formula $(\mu_1 \rightsquigarrow \phi_1) \wedge \ldots \wedge (\mu_k \rightsquigarrow \phi_k) \supset (\mu \rightsquigarrow \phi)$ is a conditional Horn formula. Then, we can prove that the validity of a conditional Horn formula is independent of whether total pre-order or partial order is used in ordered structures.

Theorem 3.6 *For any conditional Horn formula α, the following two conditions are equivalent.*

1. *α is valid under totally ordered structures.*
2. *α is valid under partially ordered structures.*

This theorem is interesting in light of axiomatizations, because in the axiomatic systems proposed in various works, non-Horn axioms are used to discriminate total pre-order cases from partial order cases.

Furthermore, we can show that the validity of a conditional Horn formula under families of ordered structures is the same as the validity under families of ordered structures with minimum.

Theorem 3.7 *For any conditional Horn formula α, the following four conditions are equivalent.*

1. *α is valid under families of totally ordered structures.*
2. *α is valid under families of totally ordered structures with minimum.*
3. *α is valid under families of partially ordered structures.*
4. *α is valid under families of partially ordered structures with minimum.*

3.3 Axiomatization

We show axiomatic systems for ordered structures and families of ordered structures (with minimum). As we see later, the axiomatic system for ordered structures is the same as the axiomatic system for families of ordered structures except for the fact that a variable of the former system ranges over a formula in either $S\text{-}Wff_{\leadsto}$ or L, but a variable of the latter system ranges over a formula in $Wff_{\leadsto}$.

First, let ζ, η and ξ be formula variables that range over formulas of L. Let γ and δ be formula variables that range over formulas of $S\text{-}Wff_{\leadsto}$. Let X, Y and Z be formula variables that range over formulas of $Wff_{\leadsto}$.

An axiomatic system TO is a set of the following axiomatic schemas and inference rules.[1]

Axiom Schemas

(PC) Truth-functional tautologies.

(ID) $\zeta \rightsquigarrow \zeta$.

(MP) $(\zeta \rightsquigarrow \eta) \supset (\zeta \supset \eta)$.

(AND) $(\zeta \rightsquigarrow \eta) \wedge (\zeta \rightsquigarrow \xi) \supset (\zeta \rightsquigarrow (\eta \wedge \xi))$.

(OR) $(\zeta \rightsquigarrow \xi) \wedge (\eta \rightsquigarrow \xi) \supset ((\zeta \vee \eta) \rightsquigarrow \xi)$.

(CE) $(\zeta \rightsquigarrow \eta) \wedge (\eta \rightsquigarrow \zeta) \wedge (\zeta \rightsquigarrow \xi) \supset (\eta \rightsquigarrow \xi)$.

(RM) $(\zeta \rightsquigarrow \xi) \wedge \neg(\zeta \rightsquigarrow \neg\eta) \supset ((\zeta \wedge \eta) \rightsquigarrow \xi)$.

Inference Rules

(Mp) From δ and $\delta \supset \gamma$ infer γ.

(RCM) From $\eta \supset \xi$ infer $(\zeta \rightsquigarrow \eta) \supset (\zeta \rightsquigarrow \xi)$.

Another axiomatic system PO is the axiomatic system obtained from TO by removing the axiomatic schema (RM).

We can show a kind of the completeness theorem.

Theorem 3.8 *A formula α in S-Wff$_{\rightsquigarrow}$ is a theorem of TO (resp. PO) if and only if α is valid under totally (resp. partially) ordered structures.*

Next, let us give an axiomatization for families of ordered structures. An axiomatic system FTO (resp. FPO) is the axiomatic system obtained from TO (resp. PO) by replacing the formula variables ζ, η, ξ, δ and γ with X, Y, Z, X and Y, respectively.

Theorem 3.9 *A formula A in Wff$_{\rightsquigarrow}$ is a theorem of FTO (resp. FPO) if and only if A is valid under families of totally (resp. partially) ordered structures.*

Third, we consider an axiomatization of families of ordered structures with minimum. An axiomatic system FTOM (resp. FPOM) is the axiomatic system obtained from FTO (resp. FPO) by adding an axiomatic schema:

(CS) $(X \wedge Y) \supset (X \rightsquigarrow Y)$.

[1] TO and the other axiomatic systems given are based on the axiomatization of Bell's C logic [Bell, 1989].

Theorem 3.10 *A formula A in $Wff_{\rightsquigarrow}$ is a theorem of FTOM (resp. FPOM) if and only if A is valid under families of totally (resp. partially) ordered structures with minimum.*

For each axiomatic system above, we can construct a kind of canonical model such that a set consists of all the true formulas at a world of the model if and only if the set is a maximally consistent set under the axiomatic system. A later version of this paper will provide the details.

Finally, we consider axiomatic systems in the context of $L_{n,\rightsquigarrow}$. The following results show that we can use PO and TO even if we add restrictions to ordered structures. The results easily follow from Theorems 3.2, 3.3 and 3.8.

Corollary 3.1 *Let n be any positive integer. Then, the following hold.*

1. *A formula α in S-$Wff_{n,\rightsquigarrow}$ is a theorem of TO if and only if α is valid under finite, distinguishable, totally ordered L_n-structures.*

2. *A formula α in S-$Wff_{n,\rightsquigarrow}$ is a theorem of PO if and only if α is valid under finite, partially ordered L_n-structures.*

We note that neither TO nor PO depends on n. However, if we consider finite, *distinguishable*, partially ordered L_n-structures, then we can show that such an axiomatic system independent of n does not exist.

Theorem 3.11 *No axiomatic system $\mathcal{A}$ consisting of finite number of axiomatic schemas and inference rules satisfies the condition: for any positive integer n and for any formula α in S-$Wff_{n,\rightsquigarrow}$, the following two conditions are equivalent.*

1. *α is a theorem of $\mathcal{A}$.*

2. *α is valid under finite, distinguishable, partially ordered L_n-structures.*

Despite this theorem, Katsuno and Mendelzon [1991a] give a kind of axiomatization of finite, distinguishable, partially ordered L_n-structures in the context of knowledge base revisions. They achieve this by introducing a revision operator that corresponds to a kind of function mapping a formula to another formula.

4 Comparison

4.1 Consequence Relation

A consequence relation represents a well-behaved set of conditional assertions, where a conditional assertion $\mu \mid\!\sim \phi$ intuitively shows that μ is a good enough reason to believe ϕ. Kraus, Lehmann and Magidor [Kraus *et al.*, 1990; Lehmann, 1989] define preferential (resp. rational) consequence relations as a set of conditional assertions that is closed under a set of inference rules, **P** (resp. **R**). They also define preferential models and ranked models to discuss the semantics of the consequence relations. They show that preferential (resp. rational) consequence relations can be characterized by preferential (resp. ranked) models. It is easy to define a bijection between preferential (resp. ranked) models and partially (resp. totally) ordered structures.

By associating the connective $\rightsquigarrow$ to the connective $\mid\!\sim$, we can transform the results on TO or PO to the results on consequence relations, and vice versa. A superficial difference between consequence relations and TO or PO is the way of representations of rules and axioms. Gentzen-style rules are used in consequence relations, while a Hilbert-style axiomatization is used for ordered structures. Since a transformation between the two styles is straightforward, we can easily find a counterpart of each rule of **R** or **P** in the axioms, the inference rules, or the theorems of TO or PO. However, note that: when we consider the converse transformation, there is no counterpart to (MP) in the rules of consequence relations.

Another difference is that some logical combinations of conditional assertions are not allowed in rules of consequence relations, while all combinations are allowed in $S\text{-}Wff_{\rightsquigarrow}$. For example, $((\mu_1 \mid\!\sim \phi_1) \vee (\mu_2 \mid\!\sim \phi_2)) \supset \mu$ is not allowed, but $((\mu_1 \rightsquigarrow \phi_1) \vee (\mu_2 \rightsquigarrow \phi_2)) \supset \mu$ is allowed. Boutilier [1990a; 1990b] also extends the syntax of consequence relation so that we may use logical combinations of conditional assertion. However, his semantics of an extended formula is different from ours.

Although the above differences exist, Theorem 4.1 shows that there is no essential distinction between **R** (resp. **P**) and TO (resp. PO). For any subset Γ of $S\text{-}Wff_{\rightsquigarrow}$, we define $Con(\Gamma)$ as a set of conditional assertions such that $Con(\Gamma) = \{\mu \mid\!\sim \phi \mid \mu \rightsquigarrow \phi \in \Gamma\}$.

Theorem 4.1 *A set of conditional assertions C is a ranked (resp. preferential) consequence relation if and only if there is some deductively closed set Γ of $S\text{-}Wff_{\leadsto}$ under TO (resp. PO) such that $C = Con(\Gamma)$.*

4.2 Knowledge Base Revision

A major problem for knowledge base management is how to revise a knowledge base (KB) when new information that is inconsistent with the current KB is obtained. Alchourrón, Gärdenfors and Makinson [1985] propose rationality postulates for the revision operation. Katsuno and Mendelzon [1989; 1991a] characterize the AGM postulates in terms of minimal change with respect to an ordering among interpretations. We discuss the relationship between those works and the results on ordered structures.

Gärdenfors and his colleagues [Alchourrón *et al.*, 1985; Gärdenfors, 1988; Gärdenfors and Makinson, 1988] represent a KB as a *knowledge set*. A knowledge set is, in our context, a deductively closed set of formulas in L. Given knowledge set K and sentence μ, $K^*\mu$ is the revision of K by μ. $K^+\mu$ is the smallest deductively closed set containing K and μ. $K_\perp$ is the set consisting of all the propositional formulas. The AGM postulates consist of the following eight rules. See [Gärdenfors, 1988] for a discussion of the intuitive meaning and formal properties of these postulates.

(K*1) $K^*\mu$ is a knowledge set.

(K*2) $\mu \in K^*\mu$

(K*3) $K^*\mu \subseteq K^+\mu$

(K*4) If $\neg\mu \notin K$, then $K^+\mu \subseteq K^*\mu$

(K*5) $K^*\mu = K_\perp$ only if μ is unsatisfiable.

(K*6) If $\mu \equiv \phi$ then $K^*\mu = K^*\phi$.

(K*7) $K^*(\mu \wedge \phi) \subseteq (K^*\mu)^+\phi$

(K*8) If $\neg\phi \notin K^*\mu$ then $(K^*\mu)^+\phi \subseteq K^*(\mu \wedge \phi)$

We note that (K*3) and (K*4) imply the condition: if new knowledge μ is consistent with a knowledge set K then the revised knowledge set $K^*\mu$ is $K^+\mu$. We call this condition an *expansion condition*.

Makinson and Gärdenfors [1991] discuss similarities between these postulates and rules of consequence relations by fixing a knowledge set K and by using the transformation rule: $\phi \in K^*\mu$ iff $\mu \mid\!\sim \phi$. Since the rules of consequence relations can be translated into formulas in $S\text{-}Wff_{\leadsto}$, we can apply their discussion to the relationship between the postulates and formulas in $S\text{-}Wff_{\leadsto}$.

We give a semantic characterization of the postulates (K*1)~(K*8) by using the ordered structures. We can capture a revision operator $*$ by a collection of totally ordered structures, where a totally ordered structure is assigned to each knowledge set. The total ordered structure $\mathcal{O}_K = \langle W_K, \leq_K, V_K\rangle$ assigned to a knowledge set K must satisfy a *covering condition*[2]: for any satisfiable formula μ there is some world $w \in W_K$ such that $V_K(\mu, w) = \mathrm{T}$. The expansion condition of the postulates implies a remarkable property (the third condition of Theorem 4.2): each consistent knowledge set K consists of all the propositional formulas that are true under every minimal world of $\mathcal{O}_K$. The second condition of Theorem 4.2 corresponds to the transformation rule proposed by Makinson and Gärdenfors.

Theorem 4.2 *A revision operator $*$ satisfies (K*1)~(K*8) if and only if for each knowledge set K, there is a totally ordered structure $\mathcal{O}_K$ such that*

1. *$\mathcal{O}_K$ satisfies the covering condition,*
2. *$K^*\mu = \{\phi \mid \mathcal{O}_K \models \mu \leadsto \phi\}$,*
3. *$K = \{\phi \mid \mathcal{O}_K \models \phi\}$ if $K \neq K_\perp$.*

Katsuno and Mendelzon [1989; 1991a] consider knowledge base revision in the framework of a finitary propositional logic L_n. They represent a KB as a formula of L_n, since a computer-based KB must be finitely representable. We note that every knowledge set K (i.e., deductively closed set) can be represented in the context of L_n by a formula ψ of L_n such that $K = \{\phi \mid \psi \vdash$

[2]This property is related to (K*5).

$\phi\}$. We denote by $\psi \circ \mu$ the revision of a KB ψ by μ, where $\circ$ is a revision operator.

Katsuno and Mendelzon show the following six postulates (R1)~(R6) for a revision operator $\circ$ that are equivalent to (K*1)~(K*8).

(R1) $\psi \circ \mu$ implies μ.

(R2) If $\psi \wedge \mu$ is satisfiable then $\psi \circ \mu \equiv \psi \wedge \mu$.

(R3) If μ is satisfiable then $\psi \circ \mu$ is also satisfiable.

(R4) If $\psi_1 \equiv \psi_2$ and $\mu_1 \equiv \mu_2$ then $\psi_1 \circ \mu_1 \equiv \psi_2 \circ \mu_2$.

(R5) $(\psi \circ \mu) \wedge \phi$ implies $\psi \circ (\mu \wedge \phi)$.

(R6) If $(\psi \circ \mu) \wedge \phi$ is satisfiable then $\psi \circ (\mu \wedge \phi)$ implies $(\psi \circ \mu) \wedge \phi$.

By introducing total pre-orders among interpretations of L_n, they characterize all the revision operators satisfying (R1)~(R6) in light of minimal change with respect to the introduced total pre-orders. It is possible to rephrase the characterization in terms of finite, distinguishable, totally ordered L_n-structures, and show a theorem similar to Theorem 4.2.

Katsuno and Mendelzon [1991a] show postulates for a revision operator that is defined by minimal change with respect to *partial* orders among interpretations. The postulates consist of (R1)~(R5) and other two postulates (R7) and (R8).

(R7) If $\psi \circ \mu_1$ implies μ_2 and $\psi \circ \mu_2$ implies μ_1 then $\psi \circ \mu_1$ is equivalent to $\psi \circ \mu_2$.

(R8) $(\psi \circ \mu_1) \wedge (\psi \circ \mu_2)$ implies $\psi \circ (\mu_1 \vee \mu_2)$.

The postulate (R8) is noteworthy in light of the axiomatization of finite, distinguishable, partially ordered L_n-structures, because it is difficult to find a counterpart to (R8) in formulas of $S\text{-}Wff_{\leadsto}$.

We can show the relationship between revision operators satisfying (R1)~(R5), (R7), (R8) and finite, distinguishable, partially ordered L_n-structures. To do so, we need a restriction on partially ordered L_n-structures. We say that a partially ordered L_n-structure $\mathcal{O} = \langle W, \leq, V \rangle$ has a *minimum set* if it satisfies the condition: for any $w \in Min(W, \leq)$ and any $w' \in W$, if $w' \notin Min(W, \leq)$ then $w < w'$ holds. The reason why we must impose this condition is related to the expansion condition which requires that every model of ψ be less than any non-model with respect to the order that characterizes minimal change.

Theorem 4.3 *A revision operator $\circ$ satisfies (R1)~(R5), (R7) and (R8) if and only if for each formula ψ of L_n, there is a finite, distinguishable, partially ordered L_n-structure, $\mathcal{O}_\psi$, such that*

1. $\psi \circ \mu \equiv \bigwedge\{\phi \mid \mathcal{O}_\psi \models \mu \rightsquigarrow \phi\}$,
2. *$\mathcal{O}_\psi$ satisfies the covering condition and has a minimum set,*
3. $\psi \equiv \bigwedge\{\phi \mid \mathcal{O}_\psi \models \phi\}$ *if ψ is consistent,*
4. *if ψ and ψ_1 are logically equivalent then $\mathcal{O}_\psi$ is equal to $\mathcal{O}_{\psi_1}$.*

The fourth condition of Theorem 4.3 says that revision is independent of the syntactic representation of a KB ψ.

4.3 Knowledge Base Update

We discuss the relationship between knowledge base update and families of ordered structures with minimum. in this subsection. The revision discussed in Section 4.2 is used to modify a KB when we obtain new information about a static world, while we need another operation, *update*, to bring the KB up to date when the world described by it changes. The distinctions between update and revision are extensively discussed in [Katsuno and Mendelzon, 1991b].

Katsuno and Mendelzon propose the following postulates (U1)~(U8) for an update operator under L_n, and characterize all the update operators that satisfy the postulates in terms of partial orders among interpretations.

(U1) $\psi \diamond \mu$ implies μ.

(U2) If ψ implies μ then $\psi \diamond \mu$ is equivalent to ψ.

(U3) If both ψ and μ are satisfiable then $\psi \diamond \mu$ is also satisfiable.

(U4) If $\psi_1 \equiv \psi_2$ and $\mu_1 \equiv \mu_2$ then $\psi_1 \diamond \mu_1 \equiv \psi_2 \diamond \mu_2$.

(U5) $(\psi \diamond \mu) \wedge \phi$ implies $\psi \diamond (\mu \wedge \phi)$.

(U6) If $\psi \diamond \mu_1$ implies μ_2 and $\psi \diamond \mu_2$ implies μ_1 then $\psi \diamond \mu_1 \equiv \psi \diamond \mu_2$.

(U7) If ψ is complete then $(\psi \diamond \mu_1) \wedge (\psi \diamond \mu_2)$ implies $\psi \diamond (\mu_1 \vee \mu_2)$.

(U8) $(\psi_1 \vee \psi_2) \diamond \mu \equiv (\psi_1 \diamond \mu) \vee (\psi_2 \diamond \mu)$.

The postulates (U1)~(U8) are defined along the same lines as (R1)~(R8). However, two important differences exist; one is that (U1)~(U8) do not require the expansion condition, that is, even if a KB ψ and new information μ are consistent, the new KB $\psi \diamond \mu$ is not necessarily equivalent to $\psi \wedge \mu$. The other difference is that an update operator should satisfy a "disjunction rule" (U8) guaranteeing that each possible world of the KB is given independent consideration.

We can show that an update operator satisfying (U1)~(U8) can be identified with a family of finite, distinguishable, partially ordered L_n-structures with minimum, where an ordered structure is assigned to each interpretation of L_n.

Theorem 4.4 *An update operator $\diamond$ satisfies (U1)~(U8) if and only if there is a family of finite, distinguishable, partially ordered L_n-structures with minimum $\widehat{\mathcal{O}} = (\mathcal{O}_I)_{I \in \mathcal{I}_n}$ such that*

1. *$\mathcal{O}_I$ satisfies the covering condition,*
2. *the minimum of $\mathcal{O}_I$ is I,*
3. $\psi \diamond \mu \equiv \bigwedge\{\phi \mid \widehat{\mathcal{O}} \models \psi \supset (\mu \rightsquigarrow \phi)\}$.

Table 1: Conditional Logic and Axiomatization

Conditional Logic	Axiomatization
VW ([Lewis, 1973; Nute, 1984])	FTO
VC ([Lewis, 1973; Nute, 1984])	FTOM
SS or C ([Nute, 1984; Bell, 1989])	FPOM

4.4 Conditional Logic

The conditional logics consist of the propositional logic augmented with a conditional connective denoted by $>$. If we replace $\leadsto$ with $>$, we can regard $L_{\leadsto}$ as a language of conditional logics.

The various conditional logics are surveyed in [Nute, 1984]. Table 1 shows a correspondence between the proposed conditional logics and the axiomatization in Section 3.3.[3] For instance, the table states that FTO is an axiomatization of VW. In the logics listed in the table, $\mu > \phi$ intuitively means that ϕ is true under all the worlds that is most similar to μ.

Conditional logics that are not appeared in [Nute, 1984] are CT4 and CT4D proposed by Boutilier [1990a; 1990b]. The logics are formalized to represent and reason with "normality". The semantics of $\mu > \phi$ under CT4 (or CT4D) is that ϕ is true under the most *normal* situation where μ is true. Roughly speaking, he considers an order such that the more distant a world is from w, the more normal the world is. A later version of this paper will provide a formal analysis on the relationship between Boutilier's semantics and the notion of ordering.

5 Reciprocal Relation

5.1 Consequence Relation versus Revision

If we fix a knowledge set K, we can identify a revision operator $*$ satisfying (K*1)~(K*8) with a rational consequence relation determined by a ranked

[3] We have not found any conditional logic proposed in the literature that corresponds to FPO.

model satisfying the covering condition. The identification is established by Condition 2 of Theorem 4.2. In general, a revision operator $*$ satisfying (K*1)~(K*8) corresponds to some collection of rational consequence relations.

The discussion in Section 4, in light of **P** or PO, suggests a way of determining the rationality postulates for revision to knowledge sets that correspond to preferential consequence relations.

5.2 Consequence Relation versus Conditional Logic

Consequence relations are, in some sense, equivalent to "nesting-free" conditional logics. We can show the following theorem[4] by extending the discussion of Section 4.1 and by using Theorems 3.1 and 3.9.

Let r be a Gentzen-style rule of consequence relations such as

$$\frac{\alpha \wedge \beta \mathrel{|\!\sim} \gamma, \quad \alpha \mathrel{|\!\sim} \beta}{\alpha \mathrel{|\!\sim} \gamma}$$

Then, let $\hat{\alpha}(r)$ be a corresponding conditional formula such as

$$(((\alpha \wedge \beta) > \gamma) \wedge (\alpha > \beta)) \supset (\alpha > \gamma)).$$

Theorem 5.1 *For any rule r that holds in all rational (resp. preferential) consequence relations, the corresponding formula $\hat{\alpha}(r)$ of conditional logic is a theorem of VW (resp. FPO). Conversely, for any theorem α of VW (resp. FPO), if there is a rule r of consequence relations such that $\hat{\alpha}(r) = \alpha$* [5], *then the rule r holds in all rational (resp. preferential) consequence relations.*

5.3 Revision versus Conditional Logic

Gärdenfors [1988] investigates the relationship between the postulates for revision and the conditional logic VC in light of the Ramsey test: $\beta \in K^*\alpha$ iff $\alpha > \beta \in K$. He shows that the Ramsey test is incompatible with the postulate (K*4) (intuitively, the expansion condition) in his framework. Since

[4] Boutilier [1990a; 1990b] shows a similar theorem in terms of CT4 and CT4D.

[5] Note that the syntax of rules of consequence relations is restricted as discussed in Section 4.1; for example, neither $p \mathrel{|\!\sim} (q \mathrel{|\!\sim} r)$ nor $r \wedge (p \mathrel{|\!\sim} q)$ is allowed.

he considers a knowledge set constructed from formulas of $L_>$ whereas we use a knowledge set as a set of propositional formulas, we can not translate his results into our context.

A revision operator satisfying (K*1)~(K*8) is equivalent to some collection of totally ordered structures. However, the semantics of VC is determined by a family of totally ordered structures with minimum. The two differences, i.e., having minimum or not and different types of collection, suggest an incompatibility result similar to the incompatibility obtained by Gärdenfors.

5.4 Update versus Conditional Logic

The update operators satisfying (U1)~(U8) are characterized by families of finite, distinguishable, partially ordered L_π structures with minimum. The semantics of the conditional logic SS (or equivalently C) is determined by families of partially ordered structures with minimum. We can expect more similarities of update to conditional logic than those of revision from the above two facts. To develop the correspondence, we must define an update operator in the context of knowledge sets, and find postulates for the update that correspond to (U1)~(U8).

Grahne [1991] proposes a conditional logic VCU^2 having an update operator, and shows the Gärdenfors' incompatibility result does not hold in VCU^2.

6 Concluding Remarks

We define ordered structures and families of ordered structures (with minimum) as tools to develop a unified view of existing work on consequence relations, knowledge base revision, update and conditional logics. By using ordered structures and families of ordered structures, we can show reciprocal relations among the different approaches.

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