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LK-to-NK Transformation

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Abstract

This paper describes LK-to-NK transformation which maps a proof in LK to a proof in NK. We prove that a cut-free proof in LK is mapped to a normal proof in NK, in a sense, by the transformation. This is used to develop user oriented explanation facilities in automatic theorem provers and proof checkers.

1 Introduction

It is well known that there are theoretical similarities between the cut-elimination theorem for LJ and LK [Gen34] and the normalization theorem for NJ [Pra65]. These theorems say that every purely logical proof can be reduced to a definite, though not unique, *normal* form. The essential properties of such a normal proof means that it is not roundabout. No concepts are necessary for the proof except those, contained in its final result, whose use was essential to the proof. These facts raise the problem of the correspondence between cut-free proof in LK (and LJ) and normal proof in NK (and NJ). This paper provides a positive answer to this problem for LK and NK.

[Zuc74] describes LJ-to-NJ transformation and gives a careful and detailed analysis of the correspondence between cut-elimination and normalization for an intuitionistic system including arithmetic. Zucker's method is to define a many-to-one mapping from proofs in LJ to proofs in NJ. For any LJ-proof $\mathcal{D}_1$, we can obtain a sequence of LJ-proofs $\mathcal{D}_1, \mathcal{D}_2, \ldots$, in which every $\mathcal{D}_{i+1}$ is generated by applying the cut-elimination procedure to $\mathcal{D}_i$ and the sequence converge to cut-free proof. Zucker proves that the mapping's image of this sequence is the sequence of NJ-proofs whose convergence is a normal proof in LJ.

This paper presents the LK-to-NK transformation for the man-machine interaction in automatic theorem provers and proof checkers. From this point of view, we are interested in the correspondence between cut-free proof and normal proof rather than the correspondence between the cut-elimination procedure and the normalization procedure. First, we describe the LJ-to-NJ transformation using the notation used in [Zuc74] which is described in the appendix. This transformation is defined in a natural sense and a straightforward way. And we present LK-to-NK transformation as to inference rules using the LJ-to-NJ transformation. The idea is the transformation which maps an original LK-proof to another LK-proof to which an extended LJ-to-NJ transformation can apply. Lastly we prove a cut-free proof in LK is transformed to a NK-proof, which is normal in some sense.

2 LJ-to-NJ Transformation φ

We will define a transformation φ which maps LJ-proof to NJ-proof.

2.1 The Basic Idea

The basic idea is illustrated as follows. This is the same as [Zuc74].

$$\Gamma \overset{\mathcal{D}_1}{\rightarrow} C \overset{\varphi}{\Longrightarrow} \begin{matrix}\Gamma \\ \varphi(\mathcal{D}_1) \\ C\end{matrix}$$

2.2 Definition of φ

The definition of φ is by induction on the number of inference rules constituting the LJ proof figure.

1. initial sequent $\quad \mathcal{D} = A^i \rightarrow A \quad \varphi(\mathcal{D}) = A$

2. thinning-L $\quad \mathcal{D} = \dfrac{\Gamma \overset{\mathcal{D}_1}{\rightarrow} B}{A^\phi, \Gamma \rightarrow B} \quad \varphi(\mathcal{D}) = \begin{matrix}\Gamma \\ \varphi(\mathcal{D}_1) \\ B\end{matrix}$

3. thinning-R $\quad \mathcal{D} = \dfrac{\Gamma \overset{\mathcal{D}_1}{\rightarrow}}{\Gamma \rightarrow B} \quad \varphi(\mathcal{D}) = \dfrac{\begin{matrix}\Gamma \\ \varphi(\mathcal{D}_1) \\ \bot\end{matrix}}{B}\ \bot$

4. contraction-L $\quad \mathcal{D} = \dfrac{A^\alpha, A^\beta, \Gamma \overset{\mathcal{D}_1}{\rightarrow} B}{A^{\alpha\cup\beta}, \Gamma \rightarrow B} \quad \varphi(\mathcal{D}) = \begin{matrix}\Gamma\ A^\alpha \longleftrightarrow A^\beta \\ \varphi(\mathcal{D}_1) \\ B\end{matrix}$

5. contraction-R non existence in LJ

6. interchange-L non operation

7. interchange-R non existence in LJ

8. cut $\quad \mathcal{D} = \dfrac{\Gamma \overset{\mathcal{D}_1}{\rightarrow} A \quad A^\alpha, \Delta \overset{\mathcal{D}_2}{\rightarrow} B}{\Gamma^{\times\alpha}, \Delta \rightarrow B}$

   (a) $\alpha = \phi \quad \varphi(\mathcal{D}) = \begin{matrix}\Delta \\ \varphi(\mathcal{D}_2) \\ B\end{matrix}$

   (b) $\alpha \neq \phi \quad \varphi(\mathcal{D}) = \begin{matrix}\Gamma^{\times\alpha} & \\ \varphi(\mathcal{D}_1) & \\ A^\alpha & \Delta \\ \varphi(\mathcal{D}_2) & \\ B & \end{matrix}$

9. $\neg$-L $\quad \mathcal{D} = \dfrac{\Gamma \overset{\mathcal{D}_1}{\rightarrow} A}{\neg A^i, \Gamma \rightarrow} \quad \varphi(\mathcal{D}) = \dfrac{\begin{matrix}\Gamma \\ \varphi(\mathcal{D}_1) \\ A\end{matrix} \quad \neg A^i}{\bot}\ \neg - E$

10. $\neg$-R $\quad \mathcal{D} = \dfrac{A^\alpha, \Gamma \overset{\mathcal{D}_1}{\rightarrow}}{\Gamma \rightarrow \neg A} \quad \varphi(\mathcal{D}) = \dfrac{\begin{matrix}[A^\alpha]\ \ \Gamma \\ \varphi(\mathcal{D}_1) \\ \bot\end{matrix}}{\neg A}\ \neg - I$

11. $\wedge$-L $\quad \mathcal{D} = \dfrac{A^\alpha, \Gamma \overset{\mathcal{D}_1}{\rightarrow} C}{(A \wedge B)^\alpha, \Gamma \rightarrow C}$

(a) $\alpha = \phi$ $\qquad \varphi(\mathcal{D}) = \begin{array}{c}\Gamma\\ \varphi(\mathcal{D}_1)\\ C\end{array}$

(b) $\alpha \neq \phi$ $\qquad \varphi(\mathcal{D}) = \begin{array}{c}\dfrac{(A \wedge B)^\alpha}{A^\alpha}\ \wedge -E \quad \Gamma\\ \varphi(\mathcal{D}_1)\\ C\end{array}$

12. $\wedge$-R $\qquad \mathcal{D} = \dfrac{\Gamma^\alpha \overset{\mathcal{D}_1}{\rightarrow} A \qquad \Gamma^\beta \overset{\mathcal{D}_2}{\rightarrow} B}{\Gamma^{\alpha\cup\beta} \rightarrow A \wedge B}$ $\qquad \varphi(\mathcal{D}) = \dfrac{\begin{array}{c}\Gamma^\alpha\\ \varphi(\mathcal{D}_1)\\ A\end{array} \longleftrightarrow \begin{array}{c}\Gamma^\beta\\ \varphi(\mathcal{D}_2)\\ B\end{array}}{A \wedge B}\ \wedge -I$

13. $\vee$-L

(a) $\mathcal{D} = \dfrac{A^\phi, \Gamma^\gamma \overset{\mathcal{D}_1}{\rightarrow} C \qquad B^\phi, \Gamma^\delta \overset{\mathcal{D}_2}{\rightarrow} C}{(A \vee B)^\phi, \Gamma^\gamma \rightarrow C}$ $\qquad \varphi(\mathcal{D}) = \begin{array}{c}\Gamma^\gamma\\ \varphi(\mathcal{D}_1)\\ C\end{array}$

(b) $\mathcal{D} = \dfrac{A^\alpha, \Gamma^\gamma \overset{\mathcal{D}_1}{\rightarrow} C \qquad B^\beta, \Gamma^\delta \overset{\mathcal{D}_2}{\rightarrow} C}{(A \vee B)^i, \Gamma^{\gamma\cup\delta} \rightarrow C}$ $\qquad$ where $\alpha \neq \phi$ *or* $\beta \neq \phi$

$$\varphi(\mathcal{D}) = \dfrac{A \vee B^i \qquad \begin{array}{c}[A^\alpha]\ \Gamma^\gamma\\ \varphi(\mathcal{D}_1)\\ C\end{array} \longleftrightarrow \begin{array}{c}\Gamma^\delta\ [B^\beta]\\ \varphi(\mathcal{D}_2)\\ C\end{array}}{C}\ \vee -E$$

14. $\vee$-R $\qquad \mathcal{D} = \dfrac{\Gamma \overset{\mathcal{D}_1}{\rightarrow} A}{\Gamma \rightarrow A \vee B}$ $\qquad \varphi(\mathcal{D}) = \dfrac{\begin{array}{c}\Gamma\\ \varphi(\mathcal{D}_1)\\ A\end{array}}{A \vee B}\ \vee -I$

15. $\supset$-L $\qquad \mathcal{D} = \dfrac{\Gamma \overset{\mathcal{D}_1}{\rightarrow} A \qquad B^\beta, \Delta \overset{\mathcal{D}_2}{\rightarrow} C}{A \supset B^\beta, \Gamma^{\times\beta}, \Delta \rightarrow C}$

(a) $\beta = \phi$ $\qquad \varphi(\mathcal{D}) = \begin{array}{c}\Delta\\ \varphi(\mathcal{D}_2)\\ C\end{array}$

(b) $\beta \neq \phi$ $\qquad \varphi(\mathcal{D}) = \begin{array}{c}\dfrac{\begin{array}{c}\Gamma^{\times\beta}\\ \varphi(\mathcal{D}_1)\\ A\end{array} \qquad (A \supset B)^\beta}{B^\beta}\ \supset -E \qquad \Delta\\ \varphi(\mathcal{D}_2)\\ C\end{array}$

16. $\supset$-R $\qquad \mathcal{D} = \dfrac{A^\alpha, \Gamma \overset{\mathcal{D}_1}{\rightarrow} B}{\Gamma \rightarrow A \supset B}$ $\qquad \varphi(\mathcal{D}) = \dfrac{\begin{array}{c}[A^\alpha]\ \Gamma\\ \varphi(\mathcal{D}_1)\\ B\end{array}}{A \supset B}\ \supset -I$

17. $\forall$-L $\qquad \mathcal{D} = \dfrac{F(t)^\alpha, \Gamma \overset{\mathcal{D}_1}{\rightarrow} B}{\forall x F(x)^\alpha, \Gamma \rightarrow B}$

(a) $\alpha = \phi$ $\qquad \varphi(\mathcal{D}) = \begin{array}{c}\Gamma\\ \varphi(\mathcal{D}_1)\\ B\end{array}$

(b) $\alpha \neq \phi$ $\qquad \varphi(\mathcal{D}) = \begin{array}{c} \dfrac{\forall x F(x)^\alpha}{F(t)^\alpha}\ \forall - E \qquad \Gamma \\ \varphi(\mathcal{D}_1) \\ B \end{array}$

18. ∀-R $\qquad \mathcal{D} = \dfrac{\Gamma \overset{\mathcal{D}_1}{\rightarrow} F(a)}{\Gamma \rightarrow \forall x F(x)} \qquad \varphi(\mathcal{D}) = \dfrac{\begin{array}{c} \Gamma \\ \varphi(\mathcal{D}_1) \\ F(a) \end{array}}{\forall x F(x)}\ \forall - I$

19. ∃-L

(a) $\mathcal{D} = \dfrac{F(a)^\phi, \Gamma \overset{\mathcal{D}_1}{\rightarrow} B}{\exists x F(x)^\phi, \Gamma \rightarrow B} \qquad \varphi(\mathcal{D}) = \begin{array}{c} \Gamma \\ \varphi(\mathcal{D}_1) \\ B \end{array}$

(b) $\mathcal{D} = \dfrac{F(a)^\alpha, \Gamma \overset{\mathcal{D}_1}{\rightarrow} B}{\exists x F(x)^i, \Gamma \rightarrow B}$ $\qquad$ where $\alpha \neq \phi$

$$\varphi(\mathcal{D}) = \dfrac{\exists x F(x)^i \qquad \begin{array}{c} [F(a)^\alpha]\ \Gamma \\ \varphi(\mathcal{D}_1) \\ B \end{array}}{B}\ \exists - E$$

20. ∃-R $\qquad \mathcal{D} = \dfrac{\Gamma \overset{\mathcal{D}_1}{\rightarrow} F(t)}{\Gamma \rightarrow \exists x F(x)} \qquad \varphi(\mathcal{D}) = \dfrac{\begin{array}{c} \Gamma \\ \varphi(\mathcal{D}_1) \\ F(t) \end{array}}{\exists x F(x)}\ \exists - I$

2.3 Some comments on φ

Hypothesis on unary index In any derivation $\mathcal{D}$, all unary indices that occur in initial sequents and lower sequents of ¬-L, ∨ L and ∃-L must be distinct.

Eigenvariable problem In the case of cut and ⊃-L, $\varphi(\mathcal{D})$ may violate the eigenvariable condition in $\varphi(\mathcal{D}_2)$. The eigenvariable condition can be liberalized using the notion *pure* [Pra65]. If this more liberal condition is adopted, these $\varphi(\mathcal{D})$ do not violate eigenvariable condition. So, each $\varphi(\mathcal{D})$ is NJ-proof. However, instead of adopting this condition, we assume that every LK-proof is *regular*.

3 LK-to-NK Transformation

3.1 The Basic Idea

We translate an LK-proof to another LK-proof which satisfies the following conditions. This translation is denoted by χ.

1. The proof figure consists of two parts. One is the main-part. The other a sub-part.
2. The main part is an LJ-proof. So, the main-part can be translated to NJ-proof by φ.
3. The sub-part consists of the same type LK-proof, whose end-sequent is $\rightarrow A, \neg A$ where A is an arbitrary formula, combined with the main-part by *cut*, as follows.

$$\dfrac{\rightarrow A, \neg A \qquad \neg A^\alpha, \Gamma \overset{\mathcal{D}_1}{\rightarrow}}{\Gamma \rightarrow A}\ cut$$

This proof is translated to NK-proof as follows.

$$\dfrac{\dfrac{\begin{array}{c}[\neg A^{\alpha}] \;\; \Gamma \\ \varphi(\mathcal{D}_1) \\ \bot\end{array}}{\neg\neg A}\,\neg - I}{A}\ \textit{elimination of double negation}$$

In the following, φ includes this translation to add to the definition of φ in the previous section.

To realize the above idea, we regard an LK-proof $\Gamma \overset{\mathcal{D}}{\rightarrow} A_1, \ldots, A_n$ as an LK-proof $\neg A_1, \ldots, \neg A_{i-1}, \neg A_{i+1}, \ldots, \neg A_n, \Gamma \overset{\mathcal{D}'}{\rightarrow} A_i$. And, when necessary, we move A_i to the antecedent by cut and move $A_j (1 \leq j \leq n, j \neq i)$ to the succedent by $\neg$-L. The formula A_i, which is determined for LK-proof $\mathcal{D}$, is defined by the following function $\mathcal{F}$.

$$\mathcal{F}(A \rightarrow A) = A$$

$$\mathcal{F}\left(\frac{\Gamma \overset{\mathcal{D}_1}{\rightarrow} \Delta}{D, \Gamma \rightarrow \Delta}\right) = \mathcal{F}(\mathcal{D}_1) \qquad \mathcal{F}\left(\frac{\Gamma \overset{\mathcal{D}_1}{\rightarrow} \Delta}{\Gamma \rightarrow \Delta, D}\right) = D$$

$$\mathcal{F}\left(\frac{D, D, \Gamma \overset{\mathcal{D}_1}{\rightarrow} \Delta}{D, \Gamma \rightarrow \Delta}\right) = \mathcal{F}(\mathcal{D}_1) \qquad \mathcal{F}\left(\frac{\Gamma \overset{\mathcal{D}_1}{\rightarrow} \Delta, D, D}{\Gamma \rightarrow \Delta, D}\right) = \phi$$

$$\mathcal{F}\left(\frac{\Gamma, C, D, \Pi \overset{\mathcal{D}_1}{\rightarrow} \Delta}{\Gamma, D, C, \Pi \rightarrow \Delta}\right) = \mathcal{F}(\mathcal{D}_1) \qquad \mathcal{F}\left(\frac{\Gamma \overset{\mathcal{D}_1}{\rightarrow} \Delta, C, D, \Lambda}{\Gamma \rightarrow \Delta, D, C, \Lambda}\right) = \mathcal{F}(\mathcal{D}_1)$$

$$\mathcal{F}\left(\frac{\Gamma \overset{\mathcal{D}_1}{\rightarrow} \Delta, D \qquad D, \Pi \overset{\mathcal{D}_2}{\rightarrow} \Lambda}{\Gamma, \Pi \rightarrow \Delta, \Lambda}\right) = \mathcal{F}(\mathcal{D}_2)$$

$$\mathcal{F}\left(\frac{\Gamma \overset{\mathcal{D}_1}{\rightarrow} \Delta, D}{\neg D, \Gamma \rightarrow \Delta}\right) = \phi \qquad \mathcal{F}\left(\frac{D, \Gamma \overset{\mathcal{D}_1}{\rightarrow} \Delta}{\Gamma \rightarrow \Delta, \neg D}\right) = \neg D$$

$$\mathcal{F}\left(\frac{C, \Gamma \overset{\mathcal{D}_1}{\rightarrow} \Delta}{C \wedge D, \Gamma \rightarrow \Delta}\right) = \mathcal{F}(\mathcal{D}_1) \qquad \mathcal{F}\left(\frac{\Gamma \overset{\mathcal{D}_1}{\rightarrow} \Delta, C \qquad \Gamma \overset{\mathcal{D}_2}{\rightarrow} \Delta, D}{\Gamma \rightarrow \Delta, C \wedge D}\right) = C \wedge D$$

$$\mathcal{F}\left(\frac{C, \Gamma \overset{\mathcal{D}_1}{\rightarrow} \Delta \qquad D, \Gamma \overset{\mathcal{D}_2}{\rightarrow} \Delta}{C \vee D, \Gamma \rightarrow \Delta}\right) = \begin{cases} \mathcal{F}(\mathcal{D}_1) & \text{if } \mathcal{F}(\mathcal{D}_1) = \mathcal{F}(\mathcal{D}_2) \\ \phi & \text{if } \mathcal{F}(\mathcal{D}_1) \neq \mathcal{F}(\mathcal{D}_2) \end{cases}$$

$$\mathcal{F}\left(\frac{\Gamma \overset{\mathcal{D}_1}{\rightarrow} \Delta, C}{\Gamma \rightarrow \Delta, C \vee D}\right) = C \vee D$$

$$\mathcal{F}\left(\frac{\Gamma \overset{\mathcal{D}_1}{\rightarrow} \Delta, C \qquad D, \Pi \overset{\mathcal{D}_2}{\rightarrow} \Lambda}{C \supset D, \Gamma, \Pi \rightarrow \Delta, \Lambda}\right) = \mathcal{F}(\mathcal{D}_2) \qquad \mathcal{F}\left(\frac{C, \Gamma \overset{\mathcal{D}_1}{\rightarrow} \Delta, D}{\Gamma \rightarrow \Delta, C \supset D}\right) = C \supset D$$

$$\mathcal{F}\left(\frac{F(t),\Gamma \overset{\mathcal{D}_1}{\to} \Delta}{\forall x F(x),\Gamma \to \Delta}\right) = \mathcal{F}(\mathcal{D}_1) \qquad \mathcal{F}\left(\frac{\Gamma \overset{\mathcal{D}_1}{\to} \Delta, F(a)}{\Gamma \to \Delta, \forall x F(x)}\right) = \forall x F(x)$$

$$\mathcal{F}\left(\frac{F(a),\Gamma \overset{\mathcal{D}_1}{\to} \Delta}{\exists x F(x),\Gamma \to \Delta}\right) = \mathcal{F}(\mathcal{D}_1) \qquad \mathcal{F}\left(\frac{\Gamma \overset{\mathcal{D}_1}{\to} \Delta, F(t)}{\Gamma \to \Delta, \exists x F(x)}\right) = \exists x F(x)$$

Roughly speaking, $\mathcal{F}(\mathcal{D})$ is the thinning formula or the principal formula of the R-inferences below which there are no R-inference and no $\neg$-L.

3.2 Definition of χ

In the definition of χ, let Δ, Λ denote the following.

$$\Delta = A_1, \ldots, A_n \qquad \Lambda = B_1, \ldots, B_m$$

$$\Delta' = \begin{cases} \Delta & \text{if } \mathcal{F}(\mathcal{D}_1) = \phi \\ A_1, \ldots, A_{i-1}, A_{i+1}, \ldots, A_n & \text{if } \mathcal{F}(\mathcal{D}_1) = A_i \end{cases}$$

$$\Lambda' = \begin{cases} \Lambda & \text{if } \mathcal{F}(\mathcal{D}_2) = \phi \\ B_1, \ldots, B_{i-1}, B_{i+1}, \ldots, B_m & \text{if } \mathcal{F}(\mathcal{D}_2) = B_i \end{cases}$$

When $\mathcal{F}(\mathcal{D}_1)[\mathcal{F}(\mathcal{D}_2)] = \phi$, $\neg\Delta', \Gamma \to \mathcal{F}(\mathcal{D}_1)$ $[\neg\Lambda', \Pi \to \mathcal{F}(\mathcal{D}_2)]$ is exactly same as $\neg\Delta, \Gamma \to$ $[\neg\Lambda, \Pi \to]$.

1. initial sequent $\qquad \mathcal{D} = A \to A \qquad \chi(\mathcal{D}) = A \to A$

2. thinning-L $\qquad \mathcal{D} = \dfrac{\Gamma \overset{\mathcal{D}_1}{\to} \Delta}{D, \Gamma \to \Delta}$

$$\chi(\mathcal{D}) = \cfrac{\cfrac{\neg\Delta', \Gamma \overset{\chi(\mathcal{D}_1)}{\to} \mathcal{F}(\mathcal{D}_1)}{D, \neg\Delta', \Gamma \to \mathcal{F}(\mathcal{D}_1)}\ thinning-L}{\cfrac{\underline{several\ interchanges}}{\neg\Delta', D, \Gamma \to \mathcal{F}(\mathcal{D}_1)}}$$

3. thinning-R $\qquad \mathcal{D} = \dfrac{\Gamma \overset{\mathcal{D}_1}{\to} \Delta}{\Gamma \to \Delta, D}$

   (a) $\mathcal{F}(\mathcal{D}_1) = \phi \qquad \chi(\mathcal{D}) = \dfrac{\neg\Delta, \Gamma \overset{\chi(\mathcal{D}_1)}{\to}}{\neg\Delta, \Gamma \to D}\ thinning-R$

   (b) $\mathcal{F}(\mathcal{D}_1) \neq \phi \qquad \chi(\mathcal{D}) = \cfrac{\cfrac{\cfrac{\neg\Delta', \Gamma \overset{\chi(\mathcal{D}_1)}{\to} \mathcal{F}(\mathcal{D}_1)}{\neg\mathcal{F}(\mathcal{D}_1), \neg\Delta', \Gamma \to}\ \neg-L}{\underline{several\ interchanges}}}{\cfrac{\neg\Delta, \Gamma \to}{\neg\Delta, \Gamma \to D}\ thinning-R}$

4. contraction-L $\qquad \mathcal{D} = \dfrac{D, D, \Gamma \overset{\mathcal{D}_1}{\to} \Delta}{D, \Gamma \to \Delta}$

$$\chi(\mathcal{D}) = \cfrac{\cfrac{\cfrac{\neg\Delta', D, D, \Gamma \overset{\chi(\mathcal{D}_1)}{\rightarrow} \mathcal{F}(\mathcal{D}_1)}{\text{several interchanges}}}{D, D, \neg\Delta', \Gamma \rightarrow \mathcal{F}(\mathcal{D}_1)}}{\cfrac{\cfrac{D, \neg\Delta', \Gamma \rightarrow \mathcal{F}(\mathcal{D}_1)}{\text{several interchanges}}}{\neg\Delta', D, \Gamma \rightarrow \mathcal{F}(\mathcal{D}_1)}} \ contraction - L$$

5. contraction-R $\quad \mathcal{D} = \dfrac{\Gamma \overset{\mathcal{D}_1}{\rightarrow} \Delta, D, D}{\Gamma \rightarrow \Delta, D}$

(a) $\mathcal{F}(\mathcal{D}_1) = D$

$$\chi(\mathcal{D}) = \cfrac{\cfrac{\cfrac{\cfrac{\neg\Delta, \neg D, \Gamma \overset{\chi(\mathcal{D}_1)}{\rightarrow} D}{\neg D, \neg\Delta, \neg D, \Gamma \rightarrow} \neg - L}{\text{several interchanges}}}{\neg D, \neg D, \neg\Delta, \Gamma \rightarrow}}{\cfrac{\cfrac{\neg D, \neg\Delta, \Gamma \rightarrow}{\text{several interchanges}}}{\neg\Delta, \neg D, \Gamma \rightarrow}} \ contraction - L$$

(b) $\mathcal{F}(\mathcal{D}_1) \neq D$ *and* $\mathcal{F}(\mathcal{D}_1) \neq \phi$

$$\chi(\mathcal{D}) = \cfrac{\cfrac{\cfrac{\neg\Delta', \neg D, \neg D, \Gamma \overset{\chi(\mathcal{D}_1)}{\rightarrow} \mathcal{F}(\mathcal{D}_1)}{\text{several interchanges}}}{\neg D, \neg D, \neg\Delta', \Gamma \rightarrow \mathcal{F}(\mathcal{D}_1)}}{\cfrac{\cfrac{\cfrac{\neg D, \neg\Delta', \Gamma \rightarrow \mathcal{F}(\mathcal{D}_1)}{\neg\mathcal{F}(\mathcal{D}_1), \neg D, \neg\Delta', \Gamma \rightarrow} \neg - L}{\text{several interchanges}}}{\neg\Delta, \neg D, \Gamma \rightarrow}} \ contraction - L$$

(c) $\mathcal{F}(\mathcal{D}_1) = \phi$

$$\chi(\mathcal{D}) = \cfrac{\cfrac{\cfrac{\neg\Delta, \neg D, \neg D, \Gamma \rightarrow}{\text{several interchanges}}}{\neg D, \neg D, \neg\Delta, \Gamma \rightarrow}}{\cfrac{\cfrac{\neg D, \neg\Delta, \Gamma \rightarrow}{\text{several interchanges}}}{\neg\Delta, \neg D, \Gamma \rightarrow}} \ contraction \quad L$$

6. interchange-L $\quad \mathcal{D} = \dfrac{\Gamma, C, D, \Pi \overset{\mathcal{D}_1}{\rightarrow} \Delta}{\Gamma, D, C, \Pi \rightarrow \Delta}$

$$\chi(\mathcal{D}) = \frac{\neg\Delta', \Gamma, C, D, \Pi \overset{\chi(\mathcal{D}_1)}{\rightarrow} \mathcal{F}(\mathcal{D}_1)}{\neg\Delta', \Gamma, D, C, \Pi \rightarrow \mathcal{F}(\mathcal{D}_1)} \ interchange - L$$

7. interchange-R $\quad \mathcal{D} = \dfrac{\Gamma \overset{\chi(\mathcal{D}_1)}{\rightarrow} \Delta, C, D, \Lambda}{\Gamma \rightarrow \Delta, D, C, \Lambda}$

(a) $\mathcal{F}(\mathcal{D}_1) = C \qquad \chi(\mathcal{D}) = \neg\Delta, \neg D, \neg\Lambda, \Gamma \overset{\chi(\mathcal{D}_1)}{\rightarrow} C$

(b) $\mathcal{F}(\mathcal{D}_1) = D$ Similarly to $\mathcal{F}(\mathcal{D}_1) = C$

(c) $\mathcal{F}(\mathcal{D}_1) \in \Lambda$

$$\chi(\mathcal{D}) = \dfrac{\neg\Delta, \neg C, \neg D, \neg\Lambda', \Gamma \overset{\chi(\mathcal{D}_1)}{\rightarrow} \mathcal{F}(\mathcal{D}_1)}{\neg\Delta, \neg D, \neg C, \neg\Lambda', \Gamma \rightarrow \mathcal{F}(\mathcal{D}_1)}\ \textit{interchange} - L$$

(d) $\mathcal{F}(\mathcal{D}_1) \in \Delta$ Similarly to $\mathcal{F}(\mathcal{D}_1) \in \Lambda$

(e) $\mathcal{F}(\mathcal{D}_1) = \phi$ Similarly to $\mathcal{F}(\mathcal{D}_1) \in \Lambda$

8. cut

$$\mathcal{D} = \dfrac{\Gamma \overset{\mathcal{D}_1}{\rightarrow} \Delta, D \qquad D, \Pi \overset{\mathcal{D}_2}{\rightarrow} \Lambda}{\Gamma, \Pi \rightarrow \Delta, \Lambda}$$

(a) $\mathcal{F}(\mathcal{D}_1) = D$

$$\chi(\mathcal{D}) = \dfrac{\dfrac{\neg\Delta, \Gamma \overset{\chi(\mathcal{D}_1)}{\rightarrow} D \qquad \dfrac{\dfrac{\neg\Lambda', D, \Pi \overset{\chi(\mathcal{D}_2)}{\rightarrow} \mathcal{F}(\mathcal{D}_2)}{\textit{several interchanges}}}{D, \neg\Lambda', \Pi \rightarrow \mathcal{F}(\mathcal{D}_2)}}{\dfrac{\neg\Delta, \Gamma, \neg\Lambda', \Pi \rightarrow \mathcal{F}(\mathcal{D}_2)}{\dfrac{\textit{several interchanges}}{\neg\Delta, \neg\Lambda', \Gamma, \Pi \rightarrow \mathcal{F}(\mathcal{D}_2)}}}\ \textit{cut}$$

(b) $\mathcal{F}(\mathcal{D}_1) \neq D$

$$\chi(\mathcal{D}) = \dfrac{\dfrac{\dfrac{\chi(\mathcal{D}_1') \qquad \chi(\mathcal{D}_2')}{\neg\Delta, \Gamma, \neg\Lambda', \Pi \rightarrow \mathcal{F}(\mathcal{D}_2)}\ \textit{cut}}{\textit{several interchanges}}}{\neg\Delta, \neg\Lambda', \Gamma, \Pi \rightarrow \mathcal{F}(\mathcal{D}_2)}$$

$\chi(\mathcal{D}_1')$ and $\chi(\mathcal{D}_2')$ are defined as follows.

$$\chi(\mathcal{D}_1') = \dfrac{\rightarrow D, \neg D \qquad \dfrac{\dfrac{\dfrac{\neg\Delta', \neg D, \Gamma \overset{\chi(\mathcal{D}_1)}{\rightarrow} \mathcal{F}(\mathcal{D}_1)}{\neg\mathcal{F}(\mathcal{D}_1), \neg\Delta', \neg D, \Gamma \rightarrow}\ \neg - L}{\textit{several interchanges}}}{\neg D, \neg\Delta, \Gamma \rightarrow}}{\neg\Delta, \Gamma \rightarrow D}\ \textit{cut}$$

$$\chi(\mathcal{D}_2') = \dfrac{\dfrac{\neg\Delta, \Gamma, \neg\Lambda', \Pi \rightarrow \mathcal{F}(\mathcal{D}_2)}{\textit{several interchanges}}}{\neg\Delta, \neg\Lambda', \Gamma, \Pi \rightarrow \mathcal{F}(\mathcal{D}_2)}$$

9. ¬-L

$$\mathcal{D} = \dfrac{\Gamma \overset{\mathcal{D}_1}{\rightarrow} \Delta, D}{\neg D, \Gamma \rightarrow \Delta}$$

(a) $\mathcal{F}(\mathcal{D}_1) = D$

$$\chi(\mathcal{D}) = \dfrac{\dfrac{\dfrac{\neg\Delta, \Gamma \overset{\chi(\mathcal{D}_1)}{\rightarrow} D}{\neg D, \neg\Delta, \Gamma \rightarrow}\ \neg - L}{\textit{several interchanges}}}{\neg\Delta, \neg D, \Gamma \rightarrow}$$

(b) $\mathcal{F}(\mathcal{D}_1) \neq D$

$$\chi(\mathcal{D}) = \dfrac{\dfrac{\rightarrow D, \neg D \qquad \dfrac{\dfrac{\dfrac{\neg\Delta', \neg D, \Gamma \overset{\chi(\mathcal{D}_1)}{\rightarrow} \mathcal{F}(\mathcal{D}_1)}{\neg\mathcal{F}(\mathcal{D}_1), \neg\Delta', \neg D, \Gamma \rightarrow}\ \neg - L}{\textit{several interchanges}}}{\neg D, \neg\Delta, \Gamma \rightarrow}}{\neg\Delta, \Gamma \rightarrow D}\ \textit{cut}}{\dfrac{\dfrac{\neg D, \neg\Delta, \Gamma \rightarrow}{\textit{several interchanges}}}{\neg\Delta, \neg D, \Gamma \rightarrow}}\ \neg - L$$

10. ¬-R $\qquad \mathcal{D} = \dfrac{D,\Gamma \overset{\mathcal{D}_1}{\to} \Delta}{\Gamma \to \Delta, \neg D}$

(a) $\mathcal{F}(\mathcal{D}_1) = \phi \qquad \chi(\mathcal{D}) = \dfrac{\dfrac{\dfrac{\neg\Delta, D, \Gamma \overset{\chi(\mathcal{D}_1)}{\to}}{\textit{several interchanges}}}{D, \neg\Delta, \Gamma \to}}{\neg\Delta, \Gamma \to \neg D}\ \neg - R$

(b) $\mathcal{F}(\mathcal{D}_1) \neq \phi \qquad \chi(\mathcal{D}) = \dfrac{\dfrac{\dfrac{\dfrac{\neg\Delta', D, \Gamma \overset{\chi(\mathcal{D}_1)}{\to} \mathcal{F}(\mathcal{D}_1)}{\neg\mathcal{F}(\mathcal{D}_1), \neg\Delta', D, \Gamma \to}\ \neg - L}{\textit{several interchanges}}}{D, \neg\Delta, \Gamma \to}}{\neg\Delta, \Gamma \to \neg D}\ \neg - R$

11. ∧-L $\qquad \mathcal{D} = \dfrac{C,\Gamma \overset{\mathcal{D}_1}{\to} \Delta}{C \wedge D, \Gamma \to \Delta}$

$$\chi(\mathcal{D}) = \dfrac{\dfrac{\dfrac{\dfrac{\neg\Delta', C, \Gamma \overset{\chi(\mathcal{D}_1)}{\to} \mathcal{F}(\mathcal{D}_1)}{\textit{several interchanges}}}{C, \neg\Delta', \Gamma \to \mathcal{F}(\mathcal{D}_1)}}{\dfrac{\dfrac{C \wedge D, \neg\Delta', \Gamma \to \mathcal{F}(\mathcal{D}_1)}{\textit{several interchanges}}}{\neg\Delta', C \wedge D, \Gamma \to \mathcal{F}(\mathcal{D}_1)}}\ \wedge - L$$

12. ∧-R $\qquad \mathcal{D} = \dfrac{\Gamma \overset{\mathcal{D}_1}{\to} \Delta, C \qquad \Gamma \overset{\mathcal{D}_2}{\to} \Delta, D}{\Gamma \to \Delta, C \wedge D}$

$$\chi(\mathcal{D}) = \dfrac{\chi(\mathcal{D}'_1) \qquad \chi(\mathcal{D}'_2)}{\neg\Delta, \Gamma \to C \wedge D}\ \wedge - R$$

$\chi(\mathcal{D}'_1)$ and $\chi(\mathcal{D}'_2)$ are defined as follows.

(a) $\mathcal{F}(\mathcal{D}_1) = C \qquad \chi(\mathcal{D}'_1) = \neg\Delta, \Gamma \overset{\chi(\mathcal{D}_1)}{\to} C$

(b) $\mathcal{F}(\mathcal{D}_1) \neq C$ *and* $\mathcal{F}(\mathcal{D}_1) \neq \phi$

$$\chi(\mathcal{D}'_1) = \dfrac{\to C, \neg C \qquad \dfrac{\dfrac{\dfrac{\neg\Delta', \neg C, \Gamma \overset{\chi(\mathcal{D}_1)}{\to} \mathcal{F}(\mathcal{D}_1)}{\neg\mathcal{F}(\mathcal{D}_1), \neg\Delta', \neg C, \Gamma \to}\ \neg - L}{\textit{several interchanges}}}{\neg C, \neg\Delta, \Gamma \to}}{\neg D, \Gamma \to C}\ cut$$

(c) $\mathcal{F}(\mathcal{D}_1) = \phi \qquad \chi(\mathcal{D}'_1) = \dfrac{\to C, \neg C \qquad \dfrac{\dfrac{\neg\Delta, \neg C, \Gamma \overset{\chi(\mathcal{D}_1)}{\to}}{\textit{several interchanges}}}{\neg C, \neg\Delta, \Gamma \to}}{\neg D, \Gamma \to C}\ cut$

(d) $\mathcal{F}(\mathcal{D}_2) = D$ Similarly to $\mathcal{F}(\mathcal{D}_1) = C$

(e) $\mathcal{F}(\mathcal{D}_2) \neq D$ *and* $\mathcal{F}(\mathcal{D}_2) \neq \phi$ Similarly to $\mathcal{F}(\mathcal{D}_1) \neq C$ *and* $\mathcal{F}(\mathcal{D}_1) \neq \phi$

(f) $\mathcal{F}(\mathcal{D}_2) = \phi$ Similarly to $\mathcal{F}(\mathcal{D}_1) = \phi$

13. ∨-L $\quad \mathcal{D} = \dfrac{C, \Gamma \overset{\mathcal{D}_1}{\rightarrow} \Delta \qquad D, \Gamma \overset{\mathcal{D}_2}{\rightarrow} \Delta}{C \vee D, \Gamma \rightarrow \Delta}$

(a) $\mathcal{F}(\mathcal{D}_1) = \mathcal{F}(\mathcal{D}_2)$

$$\chi(\mathcal{D}) = \dfrac{\dfrac{\dfrac{\neg\Delta', C, \Gamma \overset{\chi(\mathcal{D}_1)}{\rightarrow} \mathcal{F}(\mathcal{D}_1)}{\textit{several interchanges}}}{C, \neg\Delta', \Gamma \rightarrow \mathcal{F}(\mathcal{D}_1)} \qquad \dfrac{\dfrac{\neg\Delta', D, \Gamma \overset{\chi(\mathcal{D}_2)}{\rightarrow} \mathcal{F}(\mathcal{D}_2)}{\textit{several interchanges}}}{D, \neg\Delta', \Gamma \rightarrow \mathcal{F}(\mathcal{D}_2)}}{\dfrac{\dfrac{C \vee D, \neg\Delta', \Gamma \rightarrow \mathcal{F}(\mathcal{D}_1)}{\textit{several interchanges}}}{\neg\Delta', C \vee D, \Gamma \rightarrow \mathcal{F}(\mathcal{D}_1)}} \ \wedge -L$$

(b) $\mathcal{F}(\mathcal{D}_1) \neq \mathcal{F}(\mathcal{D}_2)$

$$\chi(\mathcal{D}) = \dfrac{\dfrac{\dfrac{\dfrac{\neg\Delta', C, \Gamma \overset{\chi(\mathcal{D}_1)}{\rightarrow} \mathcal{F}(\mathcal{D}_1)}{\neg\mathcal{F}(\mathcal{D}_1), \neg\Delta', C, \Gamma \rightarrow} \neg - L}{\textit{several interchanges}}}{C, \neg\Delta, \Gamma \rightarrow} \qquad \dfrac{\dfrac{\dfrac{\neg\Delta', D, \Gamma \overset{\chi(\mathcal{D}_2)}{\rightarrow} \mathcal{F}(\mathcal{D}_2)}{\neg\mathcal{F}(\mathcal{D}_2), \neg\Delta', D, \Gamma \rightarrow} \neg - L}{\textit{several interchanges}}}{D, \neg\Delta, \Gamma \rightarrow}}{\dfrac{\dfrac{C \vee D, \neg\Delta, \Gamma \rightarrow}{\textit{several interchanges}}}{\neg\Delta, C \vee D, \Gamma \rightarrow}} \ \wedge - L$$

14. ∨-R $\quad \mathcal{D} = \dfrac{\Gamma \overset{\mathcal{D}_1}{\rightarrow} \Delta, C}{\Gamma \rightarrow \Delta, C \vee D}$

(a) $\mathcal{F}(\mathcal{D}_1) = C \qquad \chi(\mathcal{D}) = \dfrac{\neg\Delta, \Gamma \overset{\chi(\mathcal{D}_1)}{\rightarrow} C}{\neg\Delta, \Gamma \rightarrow C \vee D} \ \vee - R$

(b) $\mathcal{F}(\mathcal{D}_1) \neq C$ *and* $\mathcal{F}(\mathcal{D}_1) \neq \phi$

$$\chi(\mathcal{D}) = \dfrac{\rightarrow C, \neg C \qquad \dfrac{\dfrac{\dfrac{\neg\Delta', \neg C, \Gamma \overset{\chi(\mathcal{D}_1)}{\rightarrow} \mathcal{F}(\mathcal{D}_1)}{\neg\mathcal{F}(\mathcal{D}_1), \neg\Delta', \neg C, \Gamma \rightarrow} \neg - L}{\textit{several interchanges}}}{\neg C, \neg\Delta, \Gamma \rightarrow}}{\dfrac{\neg\Delta, \Gamma \rightarrow C}{\neg\Delta, \Gamma \rightarrow C \vee D} \ \vee - R} \ cut$$

(c) $\mathcal{F}(\mathcal{D}_1) = \phi$

$$\chi(\mathcal{D}) = \dfrac{\rightarrow C, \neg C \qquad \dfrac{\dfrac{\neg\Delta, \neg C, \Gamma \overset{\chi(\mathcal{D}_1)}{\rightarrow}}{\textit{several interchanges}}}{\neg C, \neg\Delta, \Gamma \rightarrow}}{\dfrac{\neg\Delta, \Gamma \rightarrow C}{\neg\Delta, \Gamma \rightarrow C \vee D} \ \vee - R} \ cut$$

15. ⊃-L $\quad \mathcal{D} = \dfrac{\Gamma \overset{\mathcal{D}_1}{\rightarrow} \Delta, C \qquad D, \Pi \overset{\mathcal{D}_2}{\rightarrow} \Lambda}{C \supset D, \Gamma, \Pi \rightarrow \Delta, \Lambda}$

(a) $\mathcal{F}(\mathcal{D}_1) = C$

$$\chi(\mathcal{D}) = \dfrac{\dfrac{\neg\Delta, \Gamma \overset{\chi(\mathcal{D}_1)}{\rightarrow} C \qquad \dfrac{\dfrac{\neg\Lambda', D, \Pi \overset{\chi(\mathcal{D}_2)}{\rightarrow} \mathcal{F}(\mathcal{D}_2)}{\textit{several interchanges}}}{D, \neg\Lambda', \Pi \rightarrow \mathcal{F}(\mathcal{D}_2)}}{\dfrac{C \supset D, \neg\Delta, \Gamma, \neg\Lambda', \Pi \rightarrow \mathcal{F}(\mathcal{D}_2)}{\textit{several interchanges}}}\ \supset -L}{\neg\Delta, \neg\Lambda', C \supset D, \Gamma, \Pi \rightarrow \mathcal{F}(\mathcal{D}_2)}$$

(b) $\mathcal{F}(\mathcal{D}_1) \neq C$

$$\chi(\mathcal{D}) = \dfrac{\dfrac{\chi(\mathcal{D}'_1) \qquad \chi(\mathcal{D}'_2)}{\dfrac{C \supset D, \neg\Delta, \Gamma, \neg\Lambda', \Pi \rightarrow \mathcal{F}(\mathcal{D}_2)}{\textit{several interchanges}}}\ \supset -L}{\neg\Delta, \neg\Lambda', C \supset D, \Gamma, \Pi \rightarrow \mathcal{F}(\mathcal{D}_2)}$$

$\chi(\mathcal{D}'_1)$ and $\chi(\mathcal{D}'_2)$ are defined as follows.

$$\chi(\mathcal{D}'_1) = \dfrac{\rightarrow C, \neg C \qquad \dfrac{\dfrac{\dfrac{\neg\Delta', \neg C, \Gamma \overset{\chi(\mathcal{D}_1)}{\rightarrow} \mathcal{F}(\mathcal{D}_1)}{\neg\mathcal{F}(\mathcal{D}_1), \neg\Delta', \neg C, \Gamma \rightarrow}\ \neg - L}{\textit{several interchanges}}}{\neg C, \neg\Delta, \Gamma \rightarrow}}{\neg\Delta, \Gamma \rightarrow C}\ cut$$

$$\chi(\mathcal{D}'_2) = \dfrac{\dfrac{\neg\Lambda', D, \Pi \overset{\chi(\mathcal{D}_2)}{\rightarrow} \mathcal{F}(\mathcal{D}_2)}{\textit{several interchanges}}}{D, \neg\Lambda', \Pi \rightarrow \mathcal{F}(\mathcal{D}_2)}$$

16. $\supset$-R $\qquad \mathcal{D} = \dfrac{C, \Gamma \overset{\mathcal{D}_1}{\rightarrow} \Delta, D}{\Gamma \rightarrow \Delta, C \supset D}$

(a) $\mathcal{F}(\mathcal{D}_1) = D$

$$\chi(\mathcal{D}) = \dfrac{\dfrac{\dfrac{\neg\Delta, C, \Gamma \overset{\chi(\mathcal{D}_1)}{\rightarrow} D}{\textit{several interchanges}}}{C, \neg\Delta, \Gamma \rightarrow D}}{\neg\Delta, \Gamma \rightarrow C \supset D}\ \supset -L$$

(b) $\mathcal{F}(\mathcal{D}_1) \neq D$ *and* $\mathcal{F}(\mathcal{D}_1) \neq \phi$

$$\chi(\mathcal{D}) = \dfrac{\rightarrow D, \neg D \qquad \dfrac{\dfrac{\dfrac{\neg\Delta', \neg D, C, \Gamma \overset{\chi(\mathcal{D}_1)}{\rightarrow} \mathcal{F}(\mathcal{D}_1)}{\neg\mathcal{F}(\mathcal{D}_1), \neg\Delta', \neg D, C, \Gamma \rightarrow}\ \neg - L}{\textit{several interchanges}}}{\neg D, C, \neg\Delta, \Gamma \rightarrow}}{\dfrac{C, \neg\Delta, \Gamma \rightarrow D}{\neg\Delta, \Gamma \rightarrow C \supset D}\ \supset -R}\ cut$$

(c) $\mathcal{F}(\mathcal{D}_1) = \phi$

$$\chi(\mathcal{D}) = \dfrac{\rightarrow D, \neg D \qquad \dfrac{\dfrac{\neg\Delta, \neg D, C, \Gamma \overset{\chi(\mathcal{D}_1)}{\rightarrow}}{\textit{several interchanges}}}{\neg D, C, \neg\Delta, \Gamma \rightarrow}}{\dfrac{C, \neg\Delta, \Gamma \rightarrow D}{\neg\Delta, \Gamma \rightarrow C \supset D}\ \supset -R}\ cut$$

17. $\forall$-L $\qquad \mathcal{D} = \dfrac{F(t), \Gamma \overset{\chi(\mathcal{D}_1)}{\rightarrow} \Delta}{\forall x F(x), \Gamma \rightarrow \Delta}$

$$\chi(\mathcal{D}) = \cfrac{\cfrac{\cfrac{\neg\Delta', F(t), \Gamma \overset{\chi(\mathcal{D}_1)}{\longrightarrow} \mathcal{F}(\mathcal{D}_1)}{\cfrac{\textit{several interchanges}}{F(t), \neg\Delta', \Gamma \rightarrow \mathcal{F}(\mathcal{D}_1)}}}{\forall x F(x), \neg\Delta', \Gamma \rightarrow \mathcal{F}(\mathcal{D}_1)}\ \forall - L}{\cfrac{\textit{several interchanges}}{\neg\Delta', \forall x F(x), \Gamma \rightarrow \mathcal{F}(\mathcal{D}_1)}}$$

18. ∀-R $\qquad \mathcal{D} = \dfrac{\Gamma \overset{\mathcal{D}_1}{\longrightarrow} \Delta, F(a)}{\Gamma \rightarrow \Delta, \forall x F(x)}$

(a) $\mathcal{F}(\mathcal{D}_1) = F(a) \qquad \chi(\mathcal{D}) = \dfrac{\neg\Delta, \Gamma \overset{\chi(\mathcal{D}_1)}{\longrightarrow} F(a)}{\neg\Delta, \Gamma \rightarrow \forall x F(x)}\ \forall - R$

(b) $\mathcal{F}(\mathcal{D}_1) \neq F(a)$ *and* $\mathcal{F}(\mathcal{D}_1) \neq \phi$

$$\chi(\mathcal{D}) = \cfrac{\rightarrow F(a), \neg F(a) \qquad \cfrac{\cfrac{\neg\Delta', \neg F(a), \Gamma \overset{\chi(\mathcal{D}_1)}{\longrightarrow} \mathcal{F}(\mathcal{D}_1)}{\neg\mathcal{F}(\mathcal{D}_1), \neg\Delta', \neg F(a), \Gamma \rightarrow}\ \neg - L}{\cfrac{\textit{several interchanges}}{\neg F(a), \neg\Delta, \Gamma \rightarrow}}}{\cfrac{\neg\Delta, \Gamma \rightarrow F(a)}{\neg\Delta, \Gamma \rightarrow \forall x F(x)}\ \forall - R}\ cut$$

(c) $\mathcal{F}(\mathcal{D}_1) = \phi \qquad \chi(\mathcal{D}) = \cfrac{\rightarrow F(a), \neg F(a) \qquad \cfrac{\neg\Delta, \neg F(a), \Gamma \overset{\chi(\mathcal{D}_1)}{\longrightarrow}}{\cfrac{\textit{several interchanges}}{\neg F(a), \neg\Delta, \Gamma \rightarrow}}}{\cfrac{\neg\Delta, \Gamma \rightarrow F(a)}{\neg\Delta, \Gamma \rightarrow \forall x F(x)}\ \forall - R}\ cut$

19. ∃-L Similarly to ∀-L

20. ∃-R Similarly to ∀-R

3.3 An extra transformation : ψ

Applying χ to an LK-proof $\Gamma \overset{\mathcal{D}}{\longrightarrow} A_1, \ldots, A_n$, we can obtain another LK-proof whose form is either $\neg A_1, \ldots, \neg A_{i-1}, \neg A_{i+1}, \ldots, \neg A_n, \Gamma \overset{\chi(\mathcal{D})}{\longrightarrow} A_i$ (if $\mathcal{F}(\mathcal{D}) = A_i$) or $\neg A_1, \ldots, \neg A_n, \Gamma \overset{\chi(\mathcal{D})}{\longrightarrow}$ (if $\chi(\mathcal{D}) = \phi$). After applying χ, we can obtain an LK-proof whose end sequent is $\Gamma \rightarrow A_1, \ldots, A_n$, as follows.

$$\cfrac{\rightarrow A_1, \neg A_1 \qquad \neg A_1, \ldots, \neg A_{i-1}, \neg A_{i+1}, \ldots, \neg A_n, \Gamma \overset{\chi(\mathcal{D})}{\longrightarrow} A_i}{\cfrac{\neg A_2, \ldots, \neg A_{i-1}, \neg A_{i+1}, \ldots, \neg A_n, \Gamma \rightarrow A_1, A_i}{\cfrac{\textit{several cuts and interchanges}}{\Gamma \rightarrow A_1, \ldots, A_n}}}\ cut$$

$$\cfrac{\rightarrow A_1, \neg A_1 \qquad \neg A_1, \ldots, \neg A_n, \Gamma \overset{\chi(\mathcal{D})}{\longrightarrow}}{\cfrac{\neg A_2, \ldots, \neg A_n, \Gamma \rightarrow A_1}{\cfrac{\textit{several cuts and interchanges}}{\Gamma \rightarrow A_1, \ldots, A_n}}}\ cut$$

We refer to this transformation as ψ.

3.4 A Modification of χ : χ'

In the case of (9) $\neg$-L $\mathcal{F}(\mathcal{D}_1) \neq D$ of the definition of χ, $\chi(\mathcal{D})$ has redundant inferences. The right upper sequent $\neg D, \neg\Delta, \Gamma \rightarrow$ of *cut* in $\chi(\mathcal{D})$ is the same as the lower sequent of the below $\neg$-L. Inferences between these two sequents are moving $\neg D$ to the succedent by $\neg$-L and, moving D back to the antecedent by *cut*. These inferences are redundant. So we obtained another transformation χ' which is same as χ except for this case. In this case, the definition of χ' is as follows.

$$\chi'(\mathcal{D}) = \cfrac{\cfrac{\neg\Delta', \neg D, \Gamma \overset{\chi'(\mathcal{D}_1)}{\rightarrow} \mathcal{F}(\mathcal{D}_1)}{\neg\mathcal{F}(\mathcal{D}_1), \neg\Delta', \neg D, \Gamma \rightarrow} \neg - L}{\underline{\mathit{several\ interchanges}}\atop \neg\Delta, \neg D, \Gamma \rightarrow}$$

3.5 Alternatives for χ

For χ, there are some alternatives which are described below. The difference between each alternative and χ is that each alternative transformation makes more redundant inference in mapped proof than χ because it is interested in the correspondence between LK and NK rather than the transformation itself. We eliminate these redundant inferences by introducing the notion of $\mathcal{F}$.

3.5.1 Gentzen's Method

LK and NK are originally introduced in [Gen34]. Gentzen proves that these formal systems formulate the classical logic introduced by Frege, Russell and Hilbert. Gentzen gives LHK(Hilbert's formalism for classical predicate logic)-to-NK transformation, NK-to-LK transformation and LK-to-LHK transformation in order to prove equivalence of LK, NK and LHK. These transformations are introduced as extended version of LHJ-to-NJ transformation, NJ-to-LJ transformation and LJ-to-LHJ transformation.

LHK-to-NK transformation and NK-to-LK transformation are obtained straightforwardly, because each conclusion of these systems consists of one formula. On the other hand, the conclusion(the succedent) of D-formula of LK consists of an arbitrary number of formulas. So, a technique needs to be introduced for a transformation on LK. Our basic idea is similar to Gentzen's.

Gentzen gives a transformation which maps the original LK-proof to another LK-proof which is LJ-proof augmented by the proof of $\rightarrow A \vee \neg A$. The transformed LK-proof can be mapped to LHK-proof by using LJ-to-LHJ transformation.

An overview of Gentzen's transformation follows. Each inference of LK-proof is transformed according to the following rule: The upper sequents are followed by inferences of several interchanges and $\neg - L$, until all formulas other than side formulas in the succedent have been negated and brought into the antecedent. Then follows an inference of the same kind as the one just transformed. Then follow new inference $\mathcal{I}$,

$$\frac{\Gamma, \neg A \rightarrow \Delta}{\Gamma \rightarrow A, \Delta} \mathcal{I}$$

and the negated formulas are brought back into the succedent. Gentzen transforms the inference $\mathcal{I}$ into LJ inference augmented by $\rightarrow A \vee \neg A$ as an axiom.

For example, in the case of $\wedge - R$,

$$\frac{\Gamma \rightarrow \Delta, C \qquad \Gamma \rightarrow \Delta, D}{\Gamma \rightarrow \Delta, C \wedge D}$$

is transformed into:

$$\dfrac{\dfrac{\Gamma \to \Delta, C}{\neg\Delta, \Gamma \to C}\,\textit{several interchanges and } \neg - L \qquad \dfrac{\Gamma \to \Delta, D}{\neg\Delta, \Gamma \to D}\,\textit{several interchanges and } \neg - L}{\dfrac{\neg\Delta, \Gamma \to C \wedge D}{\Gamma \to \Delta, C \wedge D}\,\textit{several interchanges and } \mathcal{I}}\ \wedge - R$$

3.5.2 Prawitz's Method

In the definition of χ, we regard a sequent $\Gamma \to A_1, \ldots, A_n$ as a sequent $\neg A_1, \ldots, \neg A_{i-1}, \neg A_{i+1}, \ldots, \neg A_n, \Gamma \to A_i$. Of course, we can simply regard a sequent $\Gamma \to A_1, \ldots, A_n$ as a sequent $\neg A_1, \ldots, \neg A_{n-1}, \Gamma \to A_n$. From this view, [Pra65] describes the connection between LK and NK. Practically, these two views are equal. The difference is in the case where an original LK-proof includes redundant inferences such as the following redundant repeat of *interchange* $- R$.

$$\dfrac{\dfrac{\Gamma \overset{\mathcal{D}}{\to} A, B}{\Gamma \to B, A}}{\Gamma \to A, B}$$

This LK-proof is translated to NK-proof as follows in the new view.

$$\dfrac{\dfrac{\dfrac{\dfrac{\begin{array}{c}\Gamma \quad \overset{1}{[A]} \\ \varphi(\chi(\mathcal{D})) \\ B\end{array} \qquad \overset{2}{[\neg B]}}{\bot}\ 1}{\neg\neg A}}{A} \qquad \neg A}{\dfrac{\dfrac{\bot}{\neg\neg B}\ 2}{B}}$$

According to the definition of χ defined in the previous section, these redundant inferences are ignored, though a result proof from an automatic theorem prover may not include these redundant inferences.

3.5.3 Another Simple Method

On the other hand, more simply, we can regard a sequent $\Gamma \to A_1, \ldots, A_n$ as a sequent $\neg A_1, \ldots, \neg A_n, \Gamma \to$. With this view, χ can be redefined. In this section, this χ is referred to χ_0.

The definition of χ_0 is simpler than the definition of χ, because we need not consider each case according to whether $\mathcal{F}(\mathcal{D}_1) = \phi$, $\mathcal{F}(\mathcal{D}_1) \neq \phi$, $\mathcal{F}(\mathcal{D}_1) = \textit{side formula}$ or $\mathcal{F}(\mathcal{D}_1) \neq \textit{side formula}$.

For example, the case *cut* is as follows.

$$
\chi_0(\mathcal{D}) = \dfrac{\dfrac{\to D, \neg D \qquad \dfrac{\dfrac{\neg\Delta, \neg D, \Gamma \overset{\chi_0(\mathcal{D}_1)}{\to}}{\textit{several interchanges}}}{\neg D, \neg\Delta, \Gamma \to}}{\neg\Delta, \Gamma \to D}\,\textit{cut} \qquad \dfrac{\dfrac{\neg\Lambda, D, \Pi \overset{\chi_0(\mathcal{D}_2)}{\to}}{\textit{several interchanges}}}{D, \neg\Lambda, \Pi \to}}{\dfrac{\dfrac{\neg\Delta, \Gamma, \neg\Lambda, \Pi \to}{\textit{several interchanges}}}{\neg\Delta, \neg\Lambda, \Gamma, \Pi \to}}\,\textit{cut}
$$

Why do we adopt χ rather than χ_0? The reason is that the result of χ_0 has more redundant inferences than that of χ does. For each formula A occurring in the succedent of a sequent in LK proof, χ_0 generates $\neg$-E inference such as

$$
\dfrac{\begin{matrix}\Pi \\ A\end{matrix} \qquad \neg A}{\begin{matrix}\bot \\ \Sigma\end{matrix}}\ \neg - E
$$

This type inference is ignored by using χ. Generally, $\chi_0(\mathcal{D})$ is a more complicated NK-proof than $\chi(\mathcal{D})$. This means $\chi(\mathcal{D})$ is easier for people to read than $\chi_0(\mathcal{D})$. This is appropriate for our target.

4 Some Examples of φ and χ

Some examples for φ and χ are listed.

4.1 $\to \neg \mathbf{A} \vee \mathbf{A}$

$$
\mathcal{D} = \dfrac{\dfrac{A \to A}{A \to \neg A \vee A}}{\dfrac{\dfrac{\to \neg A \vee A, \neg A}{\to \neg A \vee A, \neg A \vee A}}{\to \neg A \vee A}}
$$

$$
\psi(\chi(\mathcal{D})) = \dfrac{\to \neg A \vee A, \neg(\neg A \vee A) \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A \to A}{A \to \neg A \vee A}}{\neg(\neg A \vee A), A \to}}{A, \neg(\neg A \vee A) \to}}{\dfrac{\neg(\neg A \vee A) \to \neg A}{\neg(\neg A \vee A) \to \neg A \vee A}}}{\dfrac{\neg(\neg A \vee A), \neg(\neg A \vee A) \to}{\neg(\neg A \vee A) \to}}}{\to \neg A \vee A}
$$

$$
\varphi(\psi(\chi(\mathcal{D}))) = \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\overset{1}{A}}{\neg A \vee A} \qquad \overset{2}{\neg(\neg A \vee A)}}{\bot}}{\neg A}\,1}{\neg A \vee A} \qquad\qquad \overset{2}{\neg(\neg A \vee A)}}{\dfrac{\dfrac{\bot}{\neg\neg(\neg A \vee A)}\,2}{\neg A \vee A}}
$$

4.2 $\mathbf{A \supset B \to \neg A \vee B}$

$$
\mathcal{D} = \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A \to A \qquad B \to B}{A \supset B, A \to B}}{A, A \supset B \to B}}{A, A \supset B \to \neg A \vee B}}{A \supset B \to \neg A \vee B, \neg A}}{A \supset B \to \neg A \vee B, \neg A \vee B}}{A \supset B \to \neg A \vee B}
$$

$$
\psi(\chi(\mathcal{D})) = \dfrac{\to \neg A \vee B, \neg(\neg A \vee B) \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A \to A \qquad B \to B}{A \supset B, A \to B}}{A, A \supset B \to B}}{A, A \supset B \to \neg A \vee B}}{\neg(\neg A \vee B), A, A \supset B \to}}{A, \neg(\neg A \vee B), A \supset B \to}}{\neg(\neg A \vee B), A \supset B \to \neg A}}{\neg(\neg A \vee B), A \supset B \to \neg A \vee B}}{\neg(\neg A \vee B), \neg(\neg A \vee B), A \supset B \to}}{\neg(\neg A \vee B), A \supset B \to}}{A \supset B \to \neg A \vee B}
$$

$$
\varphi(\psi(\chi(\mathcal{D}))) = \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\overset{1}{A} \qquad A \supset B}{B}}{\neg A \vee B} \qquad \overset{2}{\neg(\neg A \vee B)}}{\bot}}{\neg A}\,1}{\neg A \vee B} \qquad\qquad \overset{2}{\neg(\neg A \vee B)}}{\dfrac{\dfrac{\bot}{\neg\neg(\neg A \vee B)}\,2}{\neg A \vee B}}
$$

5 Cut-free Proof and Normal Proof

When we construct an NJ proof from an LJ-proof by φ, we notice the following properties[Pra65],

- Initial sequent $A \rightarrow A$ corresponds to an NJ-proof consisting of A.

As we go downward in the NJ-proof, we successively enlarge, in two directions, the corresponding LJ-proof

- When we come to applications of R-rules, we usually enlarge the corresponding NJ-proof at the bottom, applying the corresponding I-rules.
- When we come to applications of L-rules, we usually enlarge the corresponding NJ-proof at the top, applying the corresponding E-rules.

More precisely,

- There is no sub-proof[1] which is put on the major premise of each E-rule.
- Each I-rule, $\bot$-rule, and *elimination of double negation* is placed under the sub-proof.
- Each minor premise of $\neg$-E, $\vee$-E, $\supset$-E and $\exists$-E is the conclusion of the sub-proof.

These properties suggest the following theorem.

Theorem 1 Let $\mathcal{D}$ be a cut-free proof in LK, and let β be a path in $\varphi(\chi(\mathcal{D}))$, and let $\sigma_1, \sigma_2, \ldots, \sigma_n$ be the sequence of segments in β. Then there is a segment σ_i, which separates two (possibly empty) parts of β, called the E-part and the I-part of β, with the following properties.

1. Each σ_j in the E-part$(j < i)$ is a major premise of an E-rule.
2. σ_i, provided that $i \neq n$, is a premise of an I-rule or of the $\bot$-rule.
3. Each A_j $(i < j < n)$ in the I-part, except the last one, is a premise of an I-rule or *elimination of double negation*. And if *elimination of double negation* exists, it is unique and the inference $\dfrac{A_i}{A_{i+1}}$ is $\neg$-I and the inference $\dfrac{A_{i+1}}{A_{i+2}}$ is *elimination of double negation*.

Proof. By induction on the length of $\mathcal{D}$, properties 1 and 2 are proved and property 3 is partially proved. For property 3, we proved that each σ_j in the I-part, except the last one $(i < j < n)$, is a premise of an I-rule, *elimination of double negation* or the $\bot$-rule.

We assume that in the I-part there is a formula which is equal to $\bot$. Let σ_j be the first occurrence of such a formula in the branch. Then each inference $\dfrac{\sigma_k}{\sigma_{k+1}}(i \leq k < j)$ is an I-rule. This means that σ_j(that is $\bot$) has subformula(s). This is a contradiction. Then there can be no formula which is equal to $\bot$ in the I-part.

This completes the proof. $\diamond$

Corollary 1 Let $\mathcal{D}$ be a cut-free proof in LK. Then $\varphi(\chi(\mathcal{D}))$ is a normal proof in NK.

Proof. If $\varphi(\chi(\mathcal{D}))$ is not a normal proof, then there is a maximum formula in $\varphi(\chi(\mathcal{D}))$. The branch containing the maximum formula violates theorem 1. This is a contradiction. So $\varphi(\chi(\mathcal{D}))$ is a normal proof. $\diamond$

Of course, the above theorem and corollary are true on χ'. But the following theorem is true on χ', but fails on χ.

Theorem 2 Let $\mathcal{D}$ be a cut-free proof in LK. Then the consequence of each *elimination of double negation* in $\varphi(\chi'(\mathcal{D}))$ is either a premise of an I-rule or a minor premise of $\supset$-E, if the consequent is not end-formula of $\varphi(\chi'(\mathcal{D}))$.

[1] the sub-proof means $\varphi(\mathcal{D}_1)$ and $\varphi(\mathcal{D}_2)$ if they exist

Proof. An *elimination of double negation* in $\varphi(\chi'(\mathcal{D}))$ corresponds to a *cut* in $\chi'(\mathcal{D})$ as follows.

$$\dfrac{\rightarrow D, \neg D \qquad \neg D, \Gamma \overset{\chi'(\mathcal{D}')}{\rightarrow}}{\Gamma \rightarrow D}\ cut$$

This type of cut occurred in $\chi'(\mathcal{D})$ in the case $\wedge$-R, $\vee$-R, $\supset$-L, $\supset$-R, $\forall$-R and $\exists$-R. The immediate inferences below the *cut* are $\wedge$-R, $\vee$-R, $\supset$-L, $\supset$-R, $\forall$-R and $\exists$-R. This indicates that the consequence of the elimination of double negation is the premise of $\wedge$-I, $\vee$-I, $\supset$-I, $\forall$-I, $\exists$-I or the minor premise of $\supset$-E, in accordance with φ.

For example, in case of $\wedge$-R($\mathcal{F}(\mathcal{D}_1) \neq C, \mathcal{F}(\mathcal{D}_1) \neq \phi, \mathcal{F}(\mathcal{D}_2) \neq D$ and $\mathcal{F}(\mathcal{D}_2) \neq \phi$), $\varphi(\chi'(\mathcal{D})) = \dfrac{\varphi(\chi'(\mathcal{D}_1')) \qquad \varphi(\chi'(\mathcal{D}_2'))}{C \wedge D} \wedge -I$. $\varphi(\chi'(\mathcal{D}_1'))$ is as follows.

$$\varphi(\chi'(\mathcal{D}_1')) = \dfrac{\dfrac{\dfrac{\begin{matrix} & 1 & \\ \neg\Delta' & [\neg C] & \Gamma \\ & \varphi(\chi'(\mathcal{D}_1)) & \\ & \mathcal{F}(\mathcal{D}_1) & \end{matrix} \qquad \neg\mathcal{F}(\mathcal{D}_1)}{\bot}\ \neg - E}{\neg\neg C}\ 1}{C}$$

$\varphi(\chi'(\mathcal{D}_2'))$ is similar to $\varphi(\chi'(\mathcal{D}_1'))$.

So the theorem is proved. $\diamond$

6 Conclusion

In this paper, we describe the transformation which maps LK-proof to NK-proof. The idea is the transformation which maps LK-proof to another LK-proof to which an extended LJ-to-NJ transformation can be easily applied. We also prove that cut-free proof in LK is transformed to a normal proof in NK.

The transformation can be used for some explanation facilities of automatic theorem provers and some interactive proof checkers to make a proof in a semi-automatic manner employing proof procedure based on LK.

7 Acknowledgements

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A Several Concepts

This section describes several concepts and notations which need to be understood in order to read this paper. Almost all the concepts came from [Pra65] and almost all the notations from [Zuc74].

A.1 LK

LK is exactly the same as in [Gen34].

A.1.1 Regular proof

A proof in LK is called *regular*[Tak87] if it satisfies two conditions (1) all eigenvariables are distinct from one another, and (2) if a free variable a occurs as an eigenvariable in a sequent S of the proof, then a occurs only in sequents above S. When a proof $\Gamma \overset{\mathcal{D}}{\to} \Delta$ is not regular, one can construct a regular proof $\Gamma \overset{\mathcal{D}'}{\to} \Delta$.

A.1.2 Notational Conveniences

1. LK-proof $\mathcal{D}$ $\mathcal{D}, \mathcal{D}_1, \ldots$ denotes LK-proof. $\Gamma \overset{\mathcal{D}}{\to} \Delta$ denotes a proof $\mathcal{D}$ with end-sequent $\Gamma \to \Delta$. So, $\mathcal{D}$ and $\Gamma \overset{\mathcal{D}}{\to} \Delta$ denote the same proof. But, $\dfrac{\mathcal{D}}{\Gamma \to \Delta}\ I$ is different from $\mathcal{D}$. $\dfrac{\mathcal{D}}{\Gamma \to \Delta}\ I$ denotes a proof which is made up of $\mathcal{D}$ and the inference I.

A.2 NK

NK is exactly the same as in [Gen34].

A.2.1 I-rule and E-rule

I-rules are $\neg$-I, $\wedge$-I, $\vee$-I, $\supset$-I, $\forall$-I and $\exists$-I. *E-rules* are $\neg$-E, $\wedge$-E, $\vee$-E, $\supset$-E, $\forall$-E and $\exists$-E. We don't regard the $\bot$-rule and *elimination of double negation* as special inferences in either I-rules or E-rules.

A.2.2 Premise and Consequence

If $\dfrac{A_1, \ldots, A_n}{B}$ is an inference rule in NK, we call $A_1, \ldots, A_n$ the *premises* and B the *consequence* of this inference. A premise A_i is a *minor* premise, if A_i is a premise of an E-rule and does not have the terminal symbol as the elimination symbol in the E-rule. A premise that is not minor is a *major* premise.

A.2.3 Path

A *path* in an NK-proof is a sequence $A_1, A_2, \ldots, A_n$ such that

1. A_1 is the top formula on the proof that is not discharged by an application of $\vee$-E or $\exists$-E.
2. A_i ($i < n$) is not the minor premise of an application of $\supset$-E or $\neg$-E and either
   (a) A_i is not the major premise of $\vee$-E or $\exists$-E, and A_{i+1} is the formula occurrence immediately below A_i, or
   (b) A_i is the major premise of an application of $\vee$-E or $\exists$-E, and A_{i+1} is an assumption discharged in the proof by the application.

3. A_n is either a minor premise of $\supset$-E or $\neg$-E, the end-formula of the proof, or a major premise of an application of $\vee$-E or $\exists$-E such that the application does not discharge any assumptions.

A.2.4 Segment

A *segment* in an NK-proof is a sequence of $A_1, A_2, \ldots, A_n$ of consecutive formula occurrences in a thread in the NK-proof such that

1. A_1 is not the consequence of an application of $\vee$-E or $\exists$-E.

2. A_i $(i < n)$ is a minor premise of an application of $\vee$-E or $\exists$-E.

3. A_n is not the minor premise of an application of $\vee$-E or $\exists$-E.

A.2.5 Maximum Segment

A *maximum segment* is a segment that begins with a consequence of an application of an I-rule, $\perp$-rule or *elimination of double negation* and ends with a major premise of an E-rule

A.2.6 Normal Proof

An NK-proof is called *normal*, when the proof contains no maximum segment and no redundant applications of $\vee$-E or $\exists$-E. An application of $\vee$-E or $\exists$-E in an NK proof is *redundant* if it has a minor premise at which no assumption is discharged.

A.2.7 Notational Conveniences

1. Contraction in NK An NK-proof $\begin{matrix}\Gamma & A^\alpha \longleftrightarrow B^\beta \\ & \Pi\end{matrix}$ denotes NK-proof $\begin{matrix}\Gamma & A^{\alpha\cup\beta} \\ & \Pi\end{matrix}$

A.3 Notational Conveniences

1. Symbols and indices A *symbol* is a finite non-empty sequence of natural numbers. An *index* is a finite non-empty set of symbols. Symbols are denoted by $\sigma, \tau, \ldots$, and indices by $\alpha, \beta, \ldots$. An index consisting of one symbol, $\{\sigma\}$, is denoted by σ. For any number i, the index $\{i\}$ is called a *unary index*, and is denoted just by i.

   We consider the following two operations on indices.

   (a) The *union* $\alpha \cup \beta$ of two indices α, β is again an index.

   (b) The *product* of α and β is defined as follows.

   $$\alpha \times \beta \stackrel{def}{=} \{\sigma * \tau | \sigma \in \alpha, \tau \in \beta\}$$

   where $*$ denotes concatenation of sequences. For example, $\sigma * \tau =$ '1''2''12''5' when $\sigma =$ '1''2' and $\tau =$ '12''5'. Especially, $\alpha \times \phi = \phi \times \alpha = \phi$ for any indices α.

2. Indexed formulas and sequents An *indexed formula* is an ordered pair of a formula and an index. We write an indexed formula (A, α) as A^α. In the definition of the φ, Γ in a sequent $\Gamma \rightarrow A$ is sequence of indexed formula.

3. Others For any sequence of index formulas $\Gamma = A_1{}^{\alpha_1}, \ldots, A_n{}^{\alpha_n}$, $\Gamma^{\times\gamma}$ denotes the sequence of index formulas $A_1{}^{\alpha_1\times\gamma}, \ldots, A_n{}^{\alpha_n\times\gamma}$.