

## A Feature-based Constraint System for Logic Programming with Entailment

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### Abstract

This paper presents the constraint system *FT*, which we feel is an intriguing alternative to Herbrand both theoretically and practically. As does Herbrand, *FT* provides a universal data structure based on trees. However, the trees of *FT* (called feature trees) are more general than the trees of Herbrand (called constructor trees), and the constraints of *FT* are finer grained and of different expressivity. The basic notion of *FT* are functional attributes called features, which provide for record-like descriptions of data avoiding the over-specification intrinsic in Herbrand's constructor-based descriptions. The feature tree structure fixes an algebraic semantics for *FT*. We will also establish a logical semantics, which is given by three axiom schemes fixing the first-order theory *FT*.

*FT* is a constraint system for logic programming, providing a test for unsatisfiability, and a test for entailment between constraints, which is needed for advanced control mechanisms.

The two major technical contributions of this paper are (1) an incremental entailment simplification system that is proved to be sound and complete, and (2) a proof showing that *FT* satisfies the so-called "independence of negative constraints".

### 1 Introduction

An important structural property of many logic programming systems is the fact that they factorize into a constraint system and an extension facility. Colmerauer's Prolog II [8] is an early language design making explicit use of this property. CLP (Constraint Logic Programming [10]), ALPS [16], CCP (Concurrent Constraint Programming [21]), and KAP (Kernel Andorra Prolog [9]) are recent logic programming frameworks that exploit this property to its full extent by being parameterized with respect to an abstract class of constraint systems. The basic operation these frameworks

require of a constraint system is a test for unsatisfiability. ALPS, CCP, and KAP in addition require a test for entailment between constraints, which is needed for advanced control mechanisms such as delaying, coroutining, synchronisation, committed choice, and deep constraint propagation. Given this situation, constraint systems are a central issue in research on logic programming.

The constraint systems of most existing logic programming languages are variations and extensions of Herbrand [14], the constraint system underlying Prolog. The individuals of Herbrand are trees corresponding to ground terms, and the atomic constraints are equations between terms. Seen from the perspective of programming, Herbrand provides a universal data structure as a logical system.

This paper presents a constraint system *FT*, which we feel is an intriguing alternative to Herbrand both theoretically and practically. As does Herbrand, *FT* provides a universal data structure based on trees. However, the trees of *FT* (called feature trees) are more general than the trees of Herbrand (called constructor trees), and the constraints of *FT* are finer grained and of different expressivity. The basic notion of *FT* are functional attributes called features, which provide for record-like descriptions of data avoiding the over-specification intrinsic in Herbrand's constructor-based descriptions. For the special case of constructor trees, features amount to argument selectors for constructors.

Suppose we want to say that *x* is a wine whose grape is riesling and whose color is white. To do this in Herbrand, one may write the equation

$$x = \text{wine}(\text{riesling}, \text{white}, y_1, \dots, y_n)$$

with the implicit assumption that the first argument of the constructor *wine* carries the "feature" *grape*, the second argument carries the "feature" *color*, and the remaining arguments  $y_1, \dots, y_n$  carry the remaining "features" of the chosen representation of wines. The obvious difficulty with this description is that it says more than we want to say, namely, that the constructor *wine* has  $n + 2$  arguments and that the "features" *grape* and *color* are represented as the first and the second argument.

The constraint system *FT* avoids this over-specification by allowing the description

$$x : \text{wine}[\text{grape} \Rightarrow \text{riesling}, \text{color} \Rightarrow \text{white}] \quad (1)$$

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saying that  $x$  has sort wine, its feature grape is riesling, and its feature color is white. Nothing is said about other features of  $x$ , which may or may not exist.

The individuals of *FT* are so-called feature trees, examples of which are shown in Figure 1. A feature tree is a possibly infinite tree whose nodes are labeled with symbols called sorts, and whose edges are labeled with symbols called features. The labeling with features is deterministic in that all edges departing from a node must be labeled with distinct features. Thus, every direct subtree of a feature tree can be identified by the feature labeling the edge leading to it. The constructor trees of Herbrand can be represented as feature trees whose edges are labeled with natural numbers indicating the corresponding argument positions.

All but the second and third feature tree in Figure 1 satisfy the description (1).

The constraints of *FT* are ordinary first-order formulae taken over a signature that accommodates sorts as unary and features as binary predicates. Thus the description (1) is actually syntactic sugar for the formula

$$\text{wine}(x) \wedge \exists y(\text{grape}(x, y) \wedge \text{riesling}(y)) \wedge \exists y(\text{color}(x, y) \wedge \text{white}(y)).$$

The set of all rational feature trees is made into a corresponding logical structure  $\mathcal{T}$  by letting  $A(x)$  hold iff the root of  $x$  is labeled with the sort  $A$ , and letting  $f(x, y)$  hold iff  $x$  has  $y$  as direct subtree via the feature  $f$ . The feature tree structure  $\mathcal{T}$  fixes an algebraic semantics for *FT*.

We will also establish a logical semantics, which is given by three axiom schemes fixing a first-order theory *FT*. Backofen and Smolka [6] show that  $\mathcal{T}$  is a model of *FT* and that *FT* is in fact a complete theory, which means that *FT* is exactly the theory induced by  $\mathcal{T}$ . However, we will not use the completeness result in the present paper, but show explicitly that entailment with respect to  $\mathcal{T}$  is the same as entailment with respect to *FT*.

The two major technical contributions of this paper are (1) an incremental entailment simplification system that is proved to be sound and complete, and (2) a proof showing that *FT* satisfies the so-called “independence of negative constraints” [7, 14, 15]. The incremental entailment simplification system is the prerequisite for *FT*’s use with either of the constraint programming frameworks ALPS, CCP or KAP mentioned at the beginning of this section. The independence property means among other things that negative constraints can essentially be handled through entailment simplification.

One origin of *FT* is Ait-Kaci’s  $\psi$ -term calculus [1], which is at the heart of the programming language LOGIN [3] and further extended in the language LIFE [5] with functions over feature structures thanks to a generalization of the concept of residuation of Le Fun [4]. Other precursors of *FT* are the feature descriptions found in so-called unification grammars [13, 12] developed for

natural language processing, and also the formalisms of Mukai [17, 18]. These early feature structure formalism were presented in a nonlogical form. Major steps in the process of their understanding and logical reformulation are the articles [20, 23, 11, 22]. Feature trees, the feature tree structure  $\mathcal{T}$ , and the axiomatization of  $\mathcal{T}$  were first given in [6].

The paper is organized as follows. Section 2 defines the basic notions and discusses the differences in expressivity between Herbrand and *FT*. Section 3 gives a basic simplification system that decides satisfiability of positive constraints. Section 4 is not committed to *FT* but discusses the notion of incremental entailment checking and its connection with the independence property and negation. Section 5 gives the entailment simplification system, proves it sound, complete and terminating, and also proves that *FT* satisfies the independence property.

## 2 Feature Trees and Constraints

To give a rigorous formalization of feature trees, we first fix two disjoint alphabets  $\mathcal{S}$  and  $\mathcal{F}$ , whose symbols are called *sorts* and *features*, respectively. The letters  $A, B, C$  will always denote sorts, and the letters  $f, g, h$  will always denote features. Words over  $\mathcal{F}$  are called paths. The concatenation of two paths  $v$  and  $w$  results in the path  $vw$ . The symbol  $\varepsilon$  denotes the empty path,  $v\varepsilon = \varepsilon v = v$ , and  $\mathcal{F}^*$  denotes the set of all paths.

A *tree domain* is a nonempty set  $D \subseteq \mathcal{F}^*$  that is prefix-closed, that is, if  $vw \in D$ , then  $v \in D$ . Thus, it always contains the empty path.

A *feature tree* is a mapping  $t : D \rightarrow \mathcal{S}$  from a tree domain  $D$  into the set of sorts. The paths in the domain of a feature tree represent the nodes of the tree; the empty path represents its root. The letters  $s$  and  $t$  are used to denote feature trees.

If convenient, we consider a feature tree  $t$  as a relation, i.e.,  $t \subseteq \mathcal{F}^* \times \mathcal{S}$ , and write  $(w, A) \in t$  instead of  $t(w) = A$ . As relations, i.e., as subsets of  $\mathcal{F}^* \times \mathcal{S}$ , feature trees are partially ordered by set inclusion. We say that  $s$  is *smaller than*  $t$  if  $s \subseteq t$ .

The *subtree*  $wt$  of a feature tree  $t$  at one of its nodes  $w$  is the feature tree defined by (as a relation):

$$wt := \{(v, A) \mid (wv, A) \in t\}.$$

If  $D$  is the domain of  $t$ , then the domain of  $wt$  is the set  $w^{-1}D = \{v \mid wv \in D\}$ . Thus,  $wt$  is given as the mapping  $wt : w^{-1}D \rightarrow \mathcal{S}$  defined on its domain by  $wt(v) = t(wv)$ . A feature tree  $s$  is called a *subtree* of a feature tree  $t$  if it is a subtree  $s = wt$  at one of its nodes  $w$ , and a *direct subtree* if  $w \in \mathcal{F}$ .

A feature tree  $t$  with domain  $D$  is called *rational* if (1)  $t$  has only finitely many subtrees and (2)  $t$  is finitely branching, which is: for every  $w \in D$ ,  $w\mathcal{F} \cap D = \{wf \in D \mid f \in \mathcal{F}\}$  is finite. Assuming (1), (2) is equivalent to

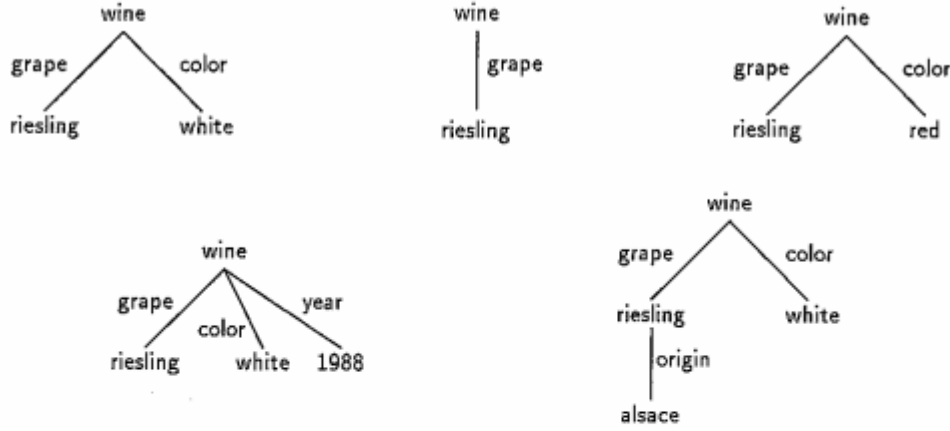


Figure 1: Examples of Feature Trees.

saying that there exist finitely many features  $f_1, \dots, f_n$  such that  $D \subseteq \{f_1, \dots, f_n\}^*$ .

Constraints over feature trees will be defined as first-order formulae. We first fix a first-order signature  $S \uplus \mathcal{F}$  by taking sorts as unary and features as binary relation symbols. Moreover, we fix an infinite alphabet of variables and adopt the convention that  $x, y, z$  always denote variables. Under this signature, every term is a variable and an atomic formula is either a feature constraint  $xfy$  ( $f(x, y)$  in standard notation), a sort constraint  $Ax$  ( $A(x)$  in standard notation), an equation  $x \doteq y$ ,  $\perp$  ("false"), or  $\top$  ("true"). Compound formulae are obtained as usual by the connectives  $\wedge, \vee, \rightarrow, \leftrightarrow, \neg$  and the quantifiers  $\exists$  and  $\forall$ . We use  $\exists\phi$  and  $\forall\phi$  to denote the existential and universal closure of a formula  $\phi$ , respectively. Moreover,  $\mathcal{V}(\phi)$  is taken to denote the set of all variables that occur free in a formula  $\phi$ . The letters  $\phi$  and  $\psi$  will always denote formulae. In the following we won't make a distinction between formulae and constraints, that is, a constraint is a formula as defined above.

$S \uplus \mathcal{F}$ -structures and validity of formulae in  $S \uplus \mathcal{F}$ -structures are defined as usual. Since we consider only  $S \uplus \mathcal{F}$ -structures in the following, we will simply speak of structures. A theory is a set of closed formulae. A model of a theory is a structure that satisfies every formulae of the theory. A formula  $\phi$  is a consequence of a theory  $T$  ( $T \models \phi$ ) if  $\forall\phi$  is valid in every model of  $T$ . A formula  $\phi$  is satisfiable in a structure  $\mathcal{A}$  if  $\exists\phi$  is valid in  $\mathcal{A}$ . Two formulae  $\phi, \psi$  are equivalent in a structure  $\mathcal{A}$  if  $\forall(\phi \leftrightarrow \psi)$  is valid in  $\mathcal{A}$ . We say that a formula  $\phi$  entails a formula  $\psi$  in a structure  $\mathcal{A}$  [theory  $T$ ] and write  $\phi \models_{\mathcal{A}} \psi$  [ $\phi \models_T \psi$ ] if  $\forall(\phi \rightarrow \psi)$  is valid in  $\mathcal{A}$  [ $\psi$  is a consequence of  $T$ ]. A theory  $T$  is complete if for every closed formula  $\phi$  either  $\phi$  or  $\neg\phi$  is a consequence of  $T$ .

The feature tree structure  $T$  is the  $S \uplus \mathcal{F}$ -structure defined as follows:

- the domain of  $T$  is the set of all rational feature

trees;

- $t \in A^T$  iff  $t(\varepsilon) = A$  ( $t$ 's root is labeled with  $A$ );
- $(s, t) \in f^T$  iff  $f \in D$ , and  $t = fs$  ( $t$  is the subtree of  $s$  at  $f$ ).

Next we discuss the expressivity of our constraints with respect to feature trees (that is, with respect to the feature tree structure  $T$ ) by means of examples. The constraint

$$\neg\exists y(xfy)$$

says that  $x$  has no subtree at  $f$ , that is, that there is no edge departing from  $x$ 's root that is labeled with  $f$ . To say that  $x$  has subtree  $y$  at path  $f_1 \dots f_n$ , we can use the constraint

$$\exists z_1 \dots \exists z_{n-1}(xf_1z_1 \wedge z_1f_2z_2 \wedge \dots \wedge z_{n-1}f_ny).$$

Now let's look at statements we cannot express (more precisely, statements of whom the authors believe they cannot be expressed). One simple unexpressible statement is " $y$  is a subtree of  $x$ " (that is, " $\exists w: y = wx$ "). Moreover, we cannot express that  $x$  is smaller than  $y$ . Finally, if we assume that the alphabet  $\mathcal{F}$  of features is infinite, we cannot say that  $x$  has subtrees at features  $f_1, \dots, f_n$  but no subtree at any other feature. In particular, we then cannot say that  $x$  is a primitive feature tree, that is, has no proper subtree.

The theory  $FT_0$  is given by the following two axiom schemes:

$$(Ax1) \quad \forall x \forall y \forall z (xfy \wedge x fz \rightarrow y \doteq z)$$

(for every feature  $f$ )

$$(Ax2) \quad \forall x (Ax \wedge Bx \rightarrow \perp)$$

(for every two distinct sorts  $A$  and  $B$ ).

The first axiom scheme says that features are functional and the second scheme says that sorts are mutually disjoint. Clearly,  $T$  is a model of  $FT_0$ . Moreover,  $FT_0$  is

incomplete (for instance,  $\exists x(Ax)$  is valid in  $\mathcal{T}$  but invalid in other models of  $FT_0$ ). We will see in the next section that  $FT_0$  plays an important role with respect to basic constraint simplification.

Next we introduce some additional notation needed in the rest of the paper. This notation will also allow us to state a third axiom scheme that, as shown in [6], extends  $FT_0$  to a complete axiomatization of  $\mathcal{T}$ .

Throughout the paper we assume that the conjunction of formulae is an associative and commutative operator that has  $\top$  as neutral element. This means that we identify  $\phi \wedge (\psi \wedge \theta)$  with  $\theta \wedge (\psi \wedge \phi)$ , and  $\phi \wedge \top$  with  $\phi$  (but not, for example,  $xfy \wedge xfy$  with  $xfy$ ). A conjunction of atomic formulae can thus be seen as the finite multiset of these formulae, where conjunction is multiset union, and  $\top$  (the “empty conjunction”) is the empty multiset. We will write  $\psi \subseteq \phi$  (or  $\psi \in \phi$ , if  $\psi$  is an atomic formula) if there exists a formula  $\psi'$  such that  $\psi \wedge \psi' = \phi$ .

We will use an additional atomic formula  $xf\uparrow$  (“ $f$  undefined on  $x$ ”) that is taken to be equivalent to  $\neg\exists y(xfy)$ , for some variable  $y$  (other than  $x$ ).

Only for the formulation of the third axiom we introduce the notion of a *solved-clause*, which is either  $\top$  or a conjunction  $\phi$  of atomic formulae of the form  $xfy$ ,  $Ax$  or  $xf\uparrow$  such that the following conditions are satisfied:

1. if  $Ax \in \phi$  and  $Bx \in \phi$ , then  $A = B$ ;
2. if  $xfy \in \phi$  and  $xfz \in \phi$ , then  $y = z$ ;
3. if  $xfy \in \phi$ , then  $xf\uparrow \notin \phi$ .

Given a solved-clause  $\phi$ , we say that a variable  $x$  is *dependent* in  $\phi$  if  $\phi$  contains a constraint of the form  $Ax$ ,  $xfy$  or  $xf\uparrow$ , and use  $\mathcal{DV}(\phi)$  to denote the set of all variables that are dependent in  $\phi$ .

The theory  $FT$  is obtained from  $FT_0$  by adding the axiom scheme:

$$(Ax3) \quad \forall \exists X \phi \\ \text{(for every solved-clause } \phi \text{ and } X = \mathcal{DV}(\phi)\text{).}$$

**Theorem 2.1** *The feature tree structure  $\mathcal{T}$  is a model of the theory  $FT$ .*

**Proof.** We will only show that  $FT$  is a model of the third axiom. Let  $X$  be the set of dependent variables of the solved-clause  $\phi$ ,  $X = \mathcal{DV}(\phi)$ . Let  $\alpha$  be any  $\mathcal{T}$ -valuation defined on  $\mathcal{V}(\phi) - X$ ; we write the tree  $\alpha(y)$  as  $t_y$ . We will extend  $\alpha$  on  $X$  such that  $\mathcal{T}, \alpha \models \phi$ .

Given  $x \in X$ , we define the “punctual” tree  $t_x = \{(\epsilon, A)\}$ , where  $A \in \mathcal{S}$  is the sort such that  $Ax \in \phi$ , if it exists, and arbitrary, otherwise. Now we are going to use the notion of *tree sum* of Nivat [19], where  $w^{-1}t = \{(wv, A) \mid (v, A) \in t\}$  (“the tree  $t$  translated by  $w$ ”), and we define:

$$\alpha(x) = \biguplus \{w^{-1}t_y \mid x \rightsquigarrow y \text{ for some } y \in \mathcal{V}(\phi), w \in \mathcal{F}^*\}.$$

Here the “leads-to” relation  $\rightsquigarrow$  is given by:  $x \rightsquigarrow x$ , and  $x \rightsquigarrow y$  if  $x \rightsquigarrow y'$  and  $y'fy \in \phi$ , for some  $y' \in \mathcal{V}(\phi)$  and some  $f \in \mathcal{F}$ . Since

$$\alpha(x) = \bigcup \{w^{-1}\alpha(y) \mid \dots\}$$

and, for a node  $w$  of  $\alpha(x)$ ,  $w\alpha(x) = \alpha(y)$ , it follows that  $\alpha(x)$  is a rational tree and that  $\mathcal{T}, \alpha \models \phi$ .  $\square$

### 3 Basic Simplification

A *basic constraint* is either  $\perp$  or a possibly empty conjunction of atomic formulae of the form  $Ax$ ,  $xfy$ , and  $x \doteq y$ . The following five *basic simplification* rules constitute a simplification system for basic constraints, which, as we will see, decides whether a basic constraint is satisfiable in  $\mathcal{T}$ .

1.  $\frac{xfy \wedge x fz \wedge \phi}{xfz \wedge y \doteq z \wedge \phi}$
2.  $\frac{Ax \wedge Bx \wedge \phi}{\perp} \quad A \neq B$
3.  $\frac{Ax \wedge Ax \wedge \phi}{Ax \wedge \phi}$
4.  $\frac{x \doteq y \wedge \phi}{x \doteq y \wedge \phi[x \leftarrow y]} \quad x \in \mathcal{V}(\phi) \text{ and } x \neq y$
5.  $\frac{x \doteq x \wedge \phi}{\phi}$

The notation  $\phi[x \leftarrow y]$  is used to denote the formula that is obtained from  $\phi$  by replacing every occurrence of  $x$  with  $y$ . We say that a constraint  $\phi$  *simplifies to* a constraint  $\psi$  by a simplification rule  $\rho$  if  $\frac{\phi}{\psi}$  is an instance of  $\rho$ . We say that a constraint  $\phi$  *simplifies to* a constraint  $\psi$  if either  $\phi = \psi$  or  $\phi$  simplifies to  $\psi$  in finitely many steps each licensed by one of the five simplification rules given above.

**Example 3.1** We have the following basic simplification chain, leading to a solved constraint:

$$\begin{aligned} & xfu \wedge yfv \wedge Au \wedge Av \wedge z \doteq x \wedge y \doteq z \\ \Rightarrow & xfu \wedge yfv \wedge Au \wedge Av \wedge z \doteq x \wedge y \doteq x \\ \Rightarrow & xfu \wedge xfv \wedge Au \wedge Av \wedge z \doteq x \wedge y \doteq x \\ \Rightarrow & xfv \wedge Au \wedge Av \wedge u \doteq v \wedge z \doteq x \wedge y \doteq x \\ \Rightarrow & xfv \wedge Av \wedge Av \wedge u \doteq v \wedge z \doteq x \wedge y \doteq x \\ \Rightarrow & xfv \wedge Av \wedge u \doteq v \wedge z \doteq x \wedge y \doteq x \end{aligned}$$

Using the same steps up to the last one, the constraint  $xfu \wedge yfv \wedge Au \wedge Bv \wedge z \doteq x \wedge y \doteq z$  simplifies to  $\perp$  (in the last step, Rule 2 instead of Rule 3 is applied).  $\square$

**Proposition 3.2** *If the basic constraint  $\phi$  simplifies to  $\psi$ , then  $FT_0 \models \phi \leftrightarrow \psi$ .*

**Proof.** The rules 3, 4 and 5 perform equivalence transformations with respect to every structure. The rules 1 and 2 correspond exactly to the two axiom schemes of  $FT_0$  and perform equivalence transformations with respect to every model of  $FT_0$ .  $\square$

We say that a basic constraint  $\phi$  binds a variable  $x$  to  $y$  if  $x \doteq y \in \phi$  and  $x$  occurs only once in  $\phi$ . At this point it is important to note that we consider equations as ordered, that is, assume that  $x \doteq y$  is different from  $y \doteq x$  if  $x \neq y$ . We say that a variable  $x$  is *eliminated*, or *bound by  $\phi$* , if  $\phi$  binds  $x$  to some variable  $y$ .

**Proposition 3.3** *The basic simplification rules are terminating.*

**Proof.** First observe that the simplification rules don't add new variables and preserve eliminated variables. Furthermore, rule 4 increases the number of eliminated variables by one. Hence we know that if an infinite simplification chain exists, we can assume without loss of generality that it only employs the rules 1, 3 and 5. Since rule 1 decreases the number of feature constraints " $xfy$ ", which is not increased by rules 3 and 5, we know that if an infinite simplification chain exists, we can assume without loss of generality that it only employs the rules 3 and 5. Since this is clearly impossible, an infinite simplification chain cannot exist.  $\square$

A basic constraint is called *normal* if none of the five simplification rules applies to it. A constraint  $\psi$  is called a *normal form* of a basic constraint  $\phi$  if  $\phi$  can be simplified to  $\psi$  and  $\psi$  is normal. A *solved constraint* is a normal constraint that is different from  $\perp$ .

So far we know that we can compute for any basic constraint  $\phi$  a normal form  $\psi$  by applying the simplification rules as long as they are applicable. Although the normal form  $\psi$  may not be unique for  $\phi$ , we know that  $\phi$  and  $\psi$  are equivalent in every model of  $FT_0$ . It remains to show that every solved constraint is satisfiable in  $\mathcal{T}$ .

Every basic constraint  $\phi$  has a unique decomposition  $\phi = \phi_N \wedge \phi_G$  such that  $\phi_N$  is a possibly empty conjunction of equations " $x \doteq y$ " and  $\phi_G$  is a possibly empty conjunction of feature constraints " $xfy$ " and sort constraints " $Ax$ ". We call  $\phi_N$  the *normalizer* and  $\phi_G$  the *graph* of  $\phi$ .

**Proposition 3.4** *A basic constraint  $\phi \neq \perp$  is solved iff the following conditions hold:*

1. *an equation  $x \doteq y$  appears in  $\phi$  only if  $x$  is eliminated in  $\phi$ ;*
2. *the graph of  $\phi$  is a solved clause;*

3. *no primitive constraint appears more than once in  $\phi$ .*

**Proposition 3.5** *Every solved constraint is satisfiable in every model of  $FT$ .*

**Proof.** Let  $\phi$  be a solved constraint and  $\mathcal{A}$  be a model of  $FT$ . Then we know by axiom scheme  $Ax3$  that the graph  $\phi_G$  of a solved constraint  $\phi$  is satisfiable in an  $FT$ -model  $\mathcal{A}$ . A variable valuation  $\alpha$  into  $\mathcal{A}$  such that  $\mathcal{A}, \alpha \models \phi_G$  can be extended on all eliminated variables simply by  $\alpha(x) = \alpha(y)$  if  $x \doteq y \in \phi$ , such that  $\mathcal{A}, \alpha \models \phi$ .  $\square$

**Theorem 3.6** *Let  $\psi$  be a normal form of a basic constraint  $\phi$ . Then  $\phi$  is satisfiable in  $\mathcal{T}$  if and only if  $\psi \neq \perp$ .*

**Proof.** Since  $\phi$  and  $\psi$  are equivalent in every model of  $FT_0$  and  $\mathcal{T}$  is a model of  $FT_0$ , it suffices to show that  $\psi$  is satisfiable in  $\mathcal{T}$  if and only if  $\psi \neq \perp$ . To show the nontrivial direction, suppose  $\psi \neq \perp$ . Then  $\psi$  is solved and we know by the preceding proposition that  $\psi$  is satisfiable in every model of  $FT$ . Since  $\mathcal{T}$  is a model of  $FT$ , we know that  $\psi$  is satisfiable in  $\mathcal{T}$ .  $\square$

**Theorem 3.7** *For every basic constraint  $\phi$  the following statements are equivalent:*

$$\mathcal{T} \models \exists \phi \Leftrightarrow \exists \text{ model } \mathcal{A} \text{ of } FT_0: \mathcal{A} \models \exists \phi \Leftrightarrow FT \models \exists \phi$$

**Proof.** The implication  $1 \Rightarrow 2$  holds since  $\mathcal{T}$  is a model of  $FT_0$ . The implication  $3 \Rightarrow 1$  follows from the fact that  $\mathcal{T}$  is a model of  $FT$ . It remains to show that  $2 \Rightarrow 3$ .

Let  $\phi$  be satisfiable in some model of  $FT_0$ . Then we can apply the simplification rules to  $\phi$  and compute a normal form  $\psi$  such that  $\phi$  and  $\psi$  are equivalent in every model of  $FT_0$ . Hence  $\psi$  is satisfiable in some model of  $FT_0$ . Thus  $\psi \neq \perp$ , which means that  $\psi$  is solved. Hence we know by the preceding proposition that  $\psi$  is satisfiable in every model of  $FT$ . Since  $\phi$  and  $\psi$  are equivalent in every model of  $FT_0 \subseteq FT$ , we have that  $\phi$  is satisfiable in every model of  $FT$ .  $\square$

## 4 Entailment, Independence and Negation

In this section we discuss some general properties of constraint entailment. This prepares the ground for the next section, which is concerned with entailment simplification in the feature tree constraint system.

Throughout this section we assume that  $\mathcal{A}$  is a structure,  $\gamma$  and  $\phi$  are formulae that can be interpreted in  $\mathcal{A}$ , and that  $X$  is a finite set of variables.

We say that  $\gamma$  *disentails*  $\phi$  in  $\mathcal{A}$  if  $\gamma$  entails  $\neg\phi$  in  $\mathcal{A}$ . If  $\gamma$  is satisfiable in  $\mathcal{A}$ , then  $\gamma$  cannot both entail and disentail  $\exists X\phi$  in  $\mathcal{A}$ . We say that  $\gamma$  *determines*  $\phi$  in  $\mathcal{A}$  if  $\gamma$  either entails or disentails  $\phi$  in  $\mathcal{A}$ .

Given  $\gamma$ ,  $\phi$  and  $X$ , we want to determine in an *incremental* manner whether  $\gamma$  entails or disentails  $\exists X\phi$ . Typically,  $\gamma$  will *not* determine  $\exists X\phi$  when  $\exists X\phi$  is considered first, but this may change when  $\gamma$  is strengthened to  $\gamma \wedge \gamma'$ . The basic idea leading to an incremental entailment checker is to simplify  $\phi$  with respect to the *context*  $\gamma$  and the *local variables*  $X$ . Given  $\gamma$ ,  $X$  and  $\phi$ , simplification must yield a formula  $\psi$  such that

$$\gamma \models_{\mathcal{A}} \exists X\phi \leftrightarrow \exists X\psi.$$

The following facts provide some evidence that this is the right invariant for entailment simplification.

**Proposition 4.1** *Let  $\gamma \models_{\mathcal{A}} \exists X\phi \leftrightarrow \exists X\psi$ . Then:*

1.  $\gamma \models_{\mathcal{A}} \exists X\phi$  iff  $\gamma \models_{\mathcal{A}} \exists X\psi$ ;
2.  $\gamma \models_{\mathcal{A}} \neg\exists X\phi$  iff  $\gamma \models_{\mathcal{A}} \neg\exists X\psi$ ;
3. if  $\psi = \perp$ , then  $\gamma \models_{\mathcal{A}} \neg\exists X\phi$ ;
4. if  $\exists X\psi$  is valid in  $\mathcal{A}$ , then  $\gamma \models_{\mathcal{A}} \exists X\phi$ .

Statements 1 and 2 say that it doesn't matter whether entailment and disentanglement are decided for  $\phi$  or  $\psi$ . Statement 3 gives a local condition for disentanglement, and Statement 4 gives a local condition for entailment. The entailment simplification system for feature trees given in the next section will in fact decide entailment and disentanglement by simplifying such that the condition of Statement 4 is met in the case of entailment, and that the condition of Statement 3 is met in the case of disentanglement.

In practice, one can ensure by variable renaming that no variable of  $X$  occurs in  $\gamma$ . The next fact says that then it suffices if entailment simplification respects the more convenient invariant

$$\mathcal{A} \models \gamma \wedge \phi \leftrightarrow \gamma \wedge \psi.$$

This is the invariant respected by our system (cf. Proposition 5.4).

**Proposition 4.2** *Let  $X \cap \mathcal{V}(\gamma) = \emptyset$ . Then:*

1. if  $\mathcal{A} \models \gamma \wedge \phi \leftrightarrow \gamma \wedge \psi$ , then  $\gamma \models_{\mathcal{A}} \exists X\phi \leftrightarrow \exists X\psi$ ;
2.  $\gamma \models_{\mathcal{A}} \neg\exists X\phi$  iff  $\gamma \wedge \phi$  is unsatisfiable in  $\mathcal{A}$ .

That is, the conjunction  $\gamma \wedge \phi$  is satisfiable if and only if  $\gamma$  either entails  $\exists X\phi$ , or it does not determine  $\exists X\phi$ .

The so-called independence of negative constraints [7, 14, 15] is an important property of constraint systems. If it holds, simplification of conjunctions of positive and negative constraints can be reduced to entailment simplification of conjunctions of positive constraints.

To define the independence property, we assume that a constraint system is a pair consisting of a structure  $\mathcal{A}$  and a set of so-called basic constraints. From basic

constraints one can build more complex constraints using the connectives and quantifiers of predicate logic. We say that a constraint system satisfies the *independence property* if

$$\gamma \models_{\mathcal{A}} \exists X_1\phi_1 \vee \dots \vee \exists X_n\phi_n \text{ iff } \exists i: \gamma \models_{\mathcal{A}} \exists X_i\phi_i$$

for all basic constraints  $\gamma$ ,  $\phi_1, \dots, \phi_n$  and all finite sets of variables  $X_1, \dots, X_n$ .

**Proposition 4.3** *If a constraint system satisfies the independence property, then the following statements hold ( $\gamma$ ,  $\phi$  and  $\phi_1, \dots, \phi_n$  are basic constraints):*

1.  $\gamma \wedge \neg\exists X_1\phi_1 \wedge \dots \wedge \neg\exists X_n\phi_n$  unsatisfiable in  $\mathcal{A}$  iff  $\exists i: \gamma \models_{\mathcal{A}} \exists X_i\phi_i$ ;
2. if  $\gamma \wedge \neg\exists X_1\phi_1 \wedge \dots \wedge \neg\exists X_n\phi_n$  is satisfiable in  $\mathcal{A}$ , then  $\gamma \wedge \neg\exists X_1\phi_1 \wedge \dots \wedge \neg\exists X_n\phi_n \models_{\mathcal{A}} \exists X\phi$  iff  $\gamma \models_{\mathcal{A}} \exists X\phi$ .

## 5 Entailment Simplification

We now return to the feature tree constraint system. Throughout this section we assume that  $\gamma$  is a solved constraint and  $X$  is a finite set of variables not occurring in  $\gamma$ . We will call  $\gamma$  the *context*, the variables in  $X$  *local*, and all other variables *global*.

If  $T$  is a theory and  $\phi$  and  $\psi$  are possibly open formulae, we write  $\phi \models_T \psi$  (read:  $\phi$  entails  $\psi$  in  $T$ ) if  $\forall(\phi \rightarrow \psi)$  is valid in  $T$ .

**Theorem 5.1** *For every basic constraint  $\phi$ , the following equivalences hold:*

$$\gamma \models_T \neg\exists X\phi \text{ iff } \gamma \models_{FT_0} \neg\exists X\phi \text{ iff } \gamma \models_{FT} \neg\exists X\phi.$$

**Proof.** Implication "2  $\Rightarrow$  3" holds since  $FT_0 \subseteq FT$ . Implication "3  $\Rightarrow$  1" holds since  $T$  is a model of  $FT$ . To show implication "1  $\Rightarrow$  2", suppose  $\gamma \models_T \neg\exists X\phi$ . Then we know by Proposition 4.2 that  $\gamma \wedge \phi$  is unsatisfiable in  $T$ . Thus we know by Theorem 3.7 that  $\gamma \wedge \phi$  is unsatisfiable in every model of  $FT_0$ . Hence we know by Proposition 4.2 that  $\gamma \models_{FT_0} \neg\exists X\phi$ .  $\square$

For every basic constraint  $\phi$  and every variable  $x$  we define

$$\phi x := \begin{cases} y & \text{if } x \doteq y \in \phi \text{ and } x \text{ is eliminated;} \\ x & \text{otherwise.} \end{cases}$$

A basic constraint  $\phi$  is *X-oriented* if  $x \doteq y \in \phi$  always implies  $x \in X$  or  $y \notin X$ . A basic constraint  $\phi$  is *pivoted* if  $x \doteq y \in \phi$  implies that  $x$  is eliminated in  $\phi$  (and then  $y$  is a "pivot").

The following *entailment simplification rules* simplify basic constraints to basic constraints with respect to a context  $\gamma$  and local variables  $X$ .



1.  $\frac{xfu \wedge \phi}{u \doteq v \wedge \phi} \quad yfv \in \gamma \wedge \phi, \quad \phi y = x$
2.  $\frac{\phi}{\phi u \doteq \phi v \wedge \phi} \quad \begin{cases} xfu \wedge yfv \subseteq \gamma, \\ \phi x = \phi y, \quad \phi u \neq \phi v, \\ \phi \text{ } X\text{-oriented and pivoted} \end{cases}$
3.  $\frac{\phi}{\perp} \quad Ax \wedge By \subseteq \gamma \wedge \phi, \quad \phi x = \phi y, \quad A \neq B$
4.  $\frac{Ax \wedge \phi}{\phi} \quad Ay \in \gamma \wedge \phi, \quad \phi y = x$
5.  $\frac{x \doteq y \wedge \phi}{x \doteq y \wedge \phi[x \leftarrow y]} \quad \begin{cases} x \neq y, \quad x \in \mathcal{V}(\phi), \\ (x \in X \text{ or } y \notin X) \end{cases}$
6.  $\frac{x \doteq y \wedge \phi}{y \doteq x \wedge \phi} \quad x \notin X, \quad y \in X$
7.  $\frac{\phi}{\phi[x \leftarrow y]} \quad x \doteq y \in \gamma, \quad x \in \mathcal{V}(\phi)$
8.  $\frac{x \doteq x \wedge \phi}{\phi}$

We say that a basic constraint  $\phi$  *simplifies* to a constraint  $\psi$  with respect to  $\gamma$  and  $X$  if  $\phi = \psi$  or  $\phi$  simplifies to  $\psi$  in finitely many steps each licensed by one of the eight simplification rules given above. The notions of *normal* and *normal form with respect to  $\gamma$*  are defined accordingly.

**Example 5.2** Let  $\gamma = xfu \wedge yfv \wedge Au \wedge Bv$  and  $X = \{z\}$ . Then we have the following simplification chain with respect to  $\gamma$  and  $X$ :

$$\begin{array}{ll}
 x \doteq z \wedge y \doteq z & \\
 \Rightarrow_{\gamma, X} z \doteq x \wedge y \doteq z & \text{by Rule E6} \\
 \Rightarrow_{\gamma, X} z \doteq x \wedge y \doteq z & \text{by Rule E5} \\
 \Rightarrow_{\gamma, X} u \doteq v \wedge z \doteq x \wedge y \doteq z & \text{by Rule E2} \\
 \Rightarrow_{\gamma, X} \perp & \text{by Rule E3.}
 \end{array}$$

Let us now take as context  $\tilde{\gamma} = xfu \wedge yfv \wedge Au$ . Then  $\tilde{\phi} = u \doteq v \wedge z \doteq x \wedge y \doteq z$  is normal with respect to  $\tilde{\gamma}$  and  $X$ . We shall see that this normal form tells us that  $\tilde{\gamma}$  does not determine  $\tilde{\phi}$ . If  $\tilde{\gamma}$  gets strengthened either to  $\tilde{\gamma} \wedge Bv$  (as above), or to  $\tilde{\gamma} \wedge x \doteq y$ , then the strengthened context does determine: it disentails in the first and entails in the second case. The basic normal form of  $\tilde{\gamma} \wedge x \doteq y$  is  $yfu \wedge Au \wedge v \doteq u \wedge x \doteq y$ ; with respect to this context  $\tilde{\phi}$  simplifies to  $z \doteq y$ .  $\square$

In the previous example,  $\phi = z \doteq x \wedge y \doteq z$  simplifies to  $\phi_1 = u \doteq v \wedge z \doteq x \wedge y \doteq z$  with respect to  $\gamma =$

$xfu \wedge yfv \wedge Au \wedge Bv$  and  $X = \{z\}$ . This corresponds to a basic simplification as follows:

$$\begin{aligned}
 & \gamma \wedge \phi = \\
 & xfu \wedge yfv \wedge Au \wedge Bv \wedge z \doteq x \wedge y \doteq z \\
 \Rightarrow & xfu \wedge xfv \wedge Au \wedge Bv \wedge z \doteq x \wedge y \doteq z \\
 \Rightarrow & xfv \wedge Au \wedge Bv \wedge u \doteq v \wedge z \doteq x \wedge y \doteq z \\
 & = \gamma' \wedge \phi'_1
 \end{aligned}$$

We observe that  $\gamma \wedge \phi_1$  is equal to  $\gamma' \wedge \phi'_1$ , modulo renaming  $y$  by  $\phi_1 y = x$  and  $u$  by  $\phi_1 u = v$ , and modulo the repetition of  $xfv$ .

**Lemma 5.3** Let  $\phi$  simplify to  $\phi_1$  with respect to  $\gamma$  and  $X$ , not using Rule E6 (in an entailment simplification step). Then  $\gamma \wedge \phi$  simplifies to some  $\gamma' \wedge \phi'_1$  which is equal to  $\gamma \wedge \phi_1$  up to variable renaming and repetition of conjuncts.

**Proof.** Clearly, each entailment simplification rule, except for E6, corresponds directly to a basic simplification rule (namely, E1 and E2 to B1, E3 to B2, E4 to B3, E5 and E7 to B4, and E8 to B5).

If the application of the entailment simplification rule to  $\phi$  relies on a condition of the form  $\phi x = y$  or  $\phi x = \phi y$  where  $x \neq \phi x$  or  $y \neq \phi y$ , then  $x \doteq \phi x \in \phi$  or  $y \doteq \phi y \in \phi$ , and Rule B4 is first applied to  $\gamma \wedge \phi$ , eliminating  $x$  by  $\phi x$  ( $y$  by  $\phi y$ ).

When comparing  $\gamma \wedge \phi_1$  and  $\gamma' \wedge \phi'_1$ , renamings take account of these variable eliminations. Note that, if the rule applied to  $\phi$  is E2, then  $\gamma'$  has one feature constraint  $xfv$  less than  $\gamma$  — which, after renaming, has a repetition of exactly this constraint.  $\square$

**Proposition 5.4** If  $\phi$  simplifies to  $\psi$  with respect to  $\gamma$  and  $X$ , then  $\gamma \wedge \phi$  and  $\gamma \wedge \psi$  are equivalent in every model of  $FT_0$ .

**Proof.** Follows from Lemma 5.3 and Proposition 3.2.  $\square$

**Proposition 5.5** The entailment simplification rules are terminating, provided  $\gamma$  and  $X$  are fixed.

**Proof.** First we strengthen the statement by weakening the applicability conditions  $\phi y = x$  in Rules E1 and E4 to  $\phi y = \phi x$ . Then from Lemma 5.3 follows: (\*) Each entailment simplification rule applies to  $\phi_1$  with respect to  $\gamma$  and  $X$  if and only if it applies to  $\phi'_1$  with respect to  $\gamma'$  and  $X$  — except possibly for E5, when the corresponding variable has already been eliminated in an “extra” basic simplification step.

If  $\gamma'$  has one conjunct of the form  $xfu$  less than  $\gamma$ , then (\*) still holds; regarding a new application of E2 this is ensured by its (therefore so complicated...) applicability condition.

With condition (\*), it is possible to prove by induction on  $n$ : For every entailment simplification chain

$\phi, \phi_1, \dots, \phi_n$  with respect to  $\gamma$  and  $X$ , there exists a 'basic plus Rule E6' simplification chain  $\gamma \wedge \phi, \gamma_2 \wedge \phi'_1, \dots, \gamma_{n+k} \wedge \phi'_{n+k}$ , where  $k \geq 0$  is the number of "extra" variable elimination steps. Since, according to Proposition 3.3, basic simplification chains are finite, so are entailment simplification chains.  $\square$

So far we know that we can compute for any basic constraint  $\phi$  a normal form  $\psi$  with respect to  $\gamma$  and  $X$  by applying the simplification rules as long as they are applicable. Although the normal form  $\psi$  may not be unique, we know that  $\gamma \wedge \phi$  and  $\gamma \wedge \psi$  are equivalent in every model of  $FT_0$ .

**Proposition 5.6** *For every basic constraint  $\phi$  one can compute a normal form  $\psi$  with respect to  $\gamma$  and  $X$ . Every such normal form  $\psi$  satisfies:  $\gamma \models_{\mathcal{T}} \exists X \phi$  iff  $\gamma \models_{\mathcal{T}} \exists X \psi$ , and  $\gamma \models_{FT} \exists X \phi$  iff  $\gamma \models_{FT} \exists X \psi$ .*

**Proof.** Follows from Propositions 5.4, 5.5, 4.2 and 4.1.  $\square$

In the following we will show that from the entailment normal form  $\psi$  of  $\phi$  with respect to  $\gamma$  it is easy to tell whether we have entailment, disentanglement or neither. Moreover, the basic normal form of  $\gamma \wedge \phi$  is exactly  $\gamma \wedge \psi$  in the first case (and in the second, where  $\gamma \wedge \perp = \perp$ ), and "almost" in the third case (cf. Lemma 5.3).

**Proposition 5.7** *A basic constraint  $\phi \neq \perp$  is normal with respect to  $\gamma$  and  $X$  if and only if the following conditions are satisfied:*

1.  $\phi$  is solved,  $X$ -oriented, and contains no variable that is bound by  $\gamma$ ;
2. if  $\phi x = y$  and  $x f u \in \gamma$ , then  $y f v \notin \phi$  for every  $v$ ;
3. if  $\phi x = \phi y$  and  $x f u \in \gamma$  and  $y f v \in \gamma$ , then  $\phi u = \phi v$ ;
4. if  $\phi x = y$  and  $A x \in \gamma$ , then  $B y \notin \phi$  for every  $B$ ;
5. if  $\phi x = \phi y$  and  $A x \in \gamma$  and  $B y \in \gamma$ , then  $A = B$ .

**Lemma 5.8** *If  $\phi \neq \perp$  is normal with respect to  $\gamma$  and  $X$ , then  $\gamma \wedge \phi$  is satisfiable in every model of  $FT$ .*

**Proof.** Let  $\phi \neq \perp$  be normal with respect to  $\gamma$  and  $X$ . Furthermore, let  $\gamma = \gamma_N \wedge \gamma_G$  and  $\phi = \phi_N \wedge \phi_G$  be the unique decompositions in normalizer and graph. Since the variables bound by  $\gamma_N$  occur neither in  $\gamma_G$  nor in  $\phi$ , it suffices to show that  $\gamma_G \wedge \phi_N \wedge \phi_G$  is satisfiable in every model of  $FT$ .

Let  $\phi_N(\gamma_G)$  be the basic constraint that is obtained from  $\gamma_G$  by applying all bindings of  $\phi_N$ . Then  $\gamma_G \wedge \phi_N \wedge \phi_G$  is equivalent to  $\phi_N \wedge \phi_N(\gamma_G) \wedge \phi_G$  and no variable bound by  $\phi_N$  occurs in  $\phi_N(\gamma_G) \wedge \phi_G$ . Hence it suffices to show that  $\phi_N(\gamma_G) \wedge \phi_G$  is satisfiable in every model of  $FT$ . With the conditions 2-5 of the preceding proposition

it is easy to see that  $\phi_N(\gamma_G) \wedge \phi_G$  is a solved clause. Hence we know by axiom scheme Ax3 that  $\phi_N(\gamma_G) \wedge \phi_G$  is satisfiable in every model of  $FT$ .  $\square$

**Theorem 5.9 (Disentanglement)** *Let  $\psi$  be a normal form of  $\phi$  with respect to  $\gamma$  and  $X$ . Then  $\gamma \models_{\mathcal{T}} \neg \exists X \phi$  iff  $\psi = \perp$ .*

**Proof.** Suppose  $\psi = \perp$ . Then  $\gamma \models_{\mathcal{T}} \neg \exists X \psi$  and hence  $\gamma \models_{\mathcal{T}} \neg \exists X \phi$  by Proposition 5.6.

To show the other direction, suppose  $\gamma \models_{\mathcal{T}} \neg \exists X \phi$ . Then  $\gamma \models_{\mathcal{T}} \neg \exists X \psi$  by Proposition 5.6 and hence  $\gamma \wedge \psi$  unsatisfiable in  $\mathcal{T}$  by Proposition 4.2. Since  $\mathcal{T}$  is a model of  $FT$  (Theorem 2.1), we know by the preceding lemma that  $\psi = \perp$  (since  $\psi$  is assumed to be normal).  $\square$

We say that a variable  $x$  is *dependent* in a solved constraint  $\phi$  if  $\phi$  contains a constraint of the form  $Ax, xfy$  or  $x \doteq y$ . (Recall that equations are ordered; thus  $y$  is not dependent in the constraint  $x \doteq y$ .) We use  $\mathcal{DV}(\phi)$  to denote the set of all variables that are dependent in a solved constraint  $\phi$ .

In the following we will assume that the underlying signature  $\mathcal{S} \uplus \mathcal{F}$  has at least one sort and at least one feature that does not occur in the constraints under consideration. This assumption is certainly satisfied if the signature has infinitely many sorts and infinitely many features.

**Lemma 5.10 (Spiting)** *Let  $\phi_1, \dots, \phi_n$  be basic constraints different from  $\perp$ , and  $X_1, \dots, X_n$  be finite sets of variables disjoint from  $\mathcal{V}(\gamma)$ . Moreover, for every  $i = 1, \dots, n$ , let  $\phi_i$  be normal with respect to  $\gamma$  and  $X_i$ , and let  $\phi_i$  have a dependent variable that is not in  $X_i$ . Then  $\gamma \wedge \neg \exists X_1 \phi_1 \wedge \dots \wedge \neg \exists X_n \phi_n$  is satisfiable in every model of  $FT$ .*

**Proof.** Let  $\gamma = \gamma_N \wedge \gamma_G$  be the unique decomposition of  $\gamma$  into normalizer and graph. Since the variables bound by  $\gamma_N$  occur neither in  $\gamma_G$  nor in any  $\phi_i$ , it suffices to show that  $\gamma_G \wedge \neg \exists X_1 \phi_1 \wedge \dots \wedge \neg \exists X_n \phi_n$  is satisfiable in every model of  $FT$ . Thus it suffices to exhibit a solved clause  $\delta$  such that  $\gamma_G \subseteq \delta$  and, for every  $i = 1, \dots, n$ ,  $\mathcal{V}(\delta)$  is disjoint with  $X_i$  and  $\delta \wedge \phi_i$  is unsatisfiable in every model of  $FT$ .

Without loss of generality we can assume that every  $X_i$  is disjoint with  $\mathcal{V}(\gamma)$  and  $\mathcal{V}(\phi_j) - X_j$  for all  $j$ . Hence we can pick in every  $\phi_i$  a dependent variable  $x_i$  such that  $x_i \notin X_i$  for any  $j$ .

Let  $z_1, \dots, z_k$  be all variables that occur on either side of equation  $x_i \doteq y \in \phi_i$ ,  $i = 1, \dots, n$  (recall that  $x_i$  is fixed for  $i$ ). None of these variables occurs in any  $X_j$  since every  $\phi_i$  is  $X_i$ -oriented. Next we fix a feature  $g$  and a sort  $B$  such that neither occurs in  $\gamma$  or any  $\phi_i$ .

Now  $\delta$  is obtained from  $\gamma$  by adding constraints as follows: if  $Ax_i \in \phi_i$ , then add  $Bx_i$ ; if  $x_i f y \in \phi_i$ , then



add  $x_i f \uparrow$ ; to enforce that the variables  $z_1, \dots, z_k$  are pairwise distinct, add

$$z_k g z_{k-1} \wedge \dots \wedge z_2 g z_1 \wedge z_1 g \uparrow.$$

It is straightforward to verify that these additions to  $\gamma$  yield a solved clause  $\delta$  as required.  $\square$

**Proposition 5.11** *If  $\phi$  is solved and  $\mathcal{DV}(\phi) \subseteq X$ , then  $FT \models \forall \exists X \phi$ .*

**Proof.** Let  $\phi = \phi_N \wedge \phi_G$  be the decomposition of  $\phi$  in normalizer and graph. Since every variable bound by  $\phi$  is in  $X$ , it suffices to show that  $\forall \exists X \phi_G$  is a consequence of  $FT$ . This follows immediately from axiom scheme  $Ax3$  since  $\phi_G$  is a solved clause.  $\square$

**Theorem 5.12 (Entailment)** *Let  $\psi$  be a normal form of  $\phi$  with respect to  $\gamma$  and  $X$ . Then  $\gamma \models_{FT} \exists X \phi$  iff  $\psi \neq \perp$  and  $\mathcal{DV}(\psi) \subseteq X$ .*

**Proof.** Suppose  $\gamma \models_{FT} \exists X \phi$ . Then we know  $\gamma \models_{FT} \exists X \psi$  by Proposition 5.6, and thus  $\gamma \wedge \neg \exists X \psi$  is unsatisfiable in  $T$ . Since  $\gamma$  is solved, we know that  $\gamma$  is satisfiable in  $T$  and hence that  $\gamma \wedge \exists X \psi$  is satisfiable in  $T$ . Thus  $\psi \neq \perp$ . Since  $\gamma \wedge \neg \exists X \psi$  is unsatisfiable in  $T$  and  $T$  is a model of  $FT$ , we know by Lemma 5.10 that  $\mathcal{DV}(\psi) \subseteq X$ .

To show the other direction, suppose  $\psi \neq \perp$  and  $\mathcal{DV}(\psi) \subseteq X$ . Then  $FT \models \forall \exists X \psi$  by Proposition 5.11, and hence  $T \models \forall \exists X \psi$ . Thus  $\gamma \models_{FT} \exists X \psi$ , and hence  $\gamma \models_{FT} \exists X \phi$  by Proposition 5.6.  $\square$

**Theorem 5.13** *Let  $\phi$  be a basic constraint. Then  $\gamma \models_{FT} \exists X \phi$  iff  $\gamma \models_{FT} \exists X \phi$ .*

**Proof.** One direction holds since  $T$  is a model of  $FT$ . To show the other direction, suppose  $\gamma \models_{FT} \exists X \phi$ . Without loss of generality we can assume that  $\phi$  is normal with respect to  $\gamma$  and  $X$ . Hence we know by Theorem 5.12 that  $\phi \neq \perp$  and  $\mathcal{DV}(\phi) \subseteq X$ . Thus  $FT \models \forall \exists X \phi$  by Proposition 5.11 and hence  $\gamma \models_{FT} \exists X \phi$ .  $\square$

**Theorem 5.14 (Independence)** *Let  $\phi_1, \dots, \phi_n$  be basic constraints, and  $X_1, \dots, X_n$  be finite sets of variables. Then*

$$\gamma \models_{FT} \exists X_1 \phi_1 \vee \dots \vee \exists X_n \phi_n \text{ iff } \exists i: \gamma \models_{FT} \exists X_i \phi_i.$$

**Proof.** To show the nontrivial direction, suppose  $\gamma \models_{FT} \exists X_1 \phi_1 \vee \dots \vee \exists X_n \phi_n$ . Without loss of generality we can assume that, for all  $i = 1, \dots, n$ ,  $X_i$  is disjoint from  $\mathcal{V}(\gamma)$ ,  $\phi_i$  is normal with respect to  $\gamma$  and  $X_i$ , and  $\phi_i \neq \perp$ . Since  $\gamma \wedge \neg \exists X_1 \phi_1 \wedge \dots \wedge \neg \exists X_n \phi_n$  is unsatisfiable in  $T$  and  $T$  is a model of  $FT$ , we know by Lemma 5.10 that  $\mathcal{DV}(\phi_k) \subseteq X_k$  for some  $k$ . Hence  $\gamma \models_{FT} \exists X_k \phi_k$  by Theorem 5.12.  $\square$

## 6 Conclusion

We have presented a constraint system  $FT$  for logic programming providing a universal data structure based on rational feature trees.  $FT$  accommodates record-like descriptions, which we think are superior to the constructor-based descriptions of Herbrand.

The declarative semantics of  $FT$  is specified both algebraically (the feature tree structure  $T$ ) and logically (the first-order theory  $FT$  given by three axiom schemes).

The operational semantics for  $FT$  is given by an incremental constraint simplification system, which can check satisfiability of and entailment between constraints. Since  $FT$  satisfies the independence property, the simplification system can also check satisfiability of conjunctions of positive and negative constraints.

We see four directions for further research.

First,  $FT$  should be strengthened such that it subsumes the expressivity of rational constructor trees [7, 8]. As is,  $FT$  cannot express that  $x$  is a tree having direct subtrees at exactly the features  $f_1, \dots, f_n$ . It turns out that the system  $CFT$  [24] obtained from  $FT$  by adding the primitive constraint

$$x\{f_1, \dots, f_n\}$$

( $x$  has direct subtrees at exactly the features  $f_1, \dots, f_n$ ) has the same nice properties as  $FT$ . In contrast to  $FT$ ,  $CFT$  can express constructor constraints; for instance, the constructor constraint  $x \doteq A(y, z)$  can be expressed equivalently as  $Ax \wedge x\{1, 2\} \wedge x1y \wedge x2z$ , if we assume that  $A$  is a sort and the numbers 1, 2 are features.

Second, it seems attractive to extend  $FT$  such that it can accommodate a sort lattice as used in [1, 3, 4, 5, 23]. One possibility to do this is to assume a partial order  $\leq$  on sorts and replace sort constraints  $Ax$  with quasi-sort constraints  $[A]x$  whose declarative semantics is given as

$$[A]x \equiv \bigvee_{B \leq A} Bx.$$

Given the assumption that the sort ordering  $\leq$  has greatest lower bounds if lower bounds exist, it seems that the results and the simplification system given for  $FT$  carry over with minor changes.

Third, the worst-case complexity of entailment checking in  $FT$  should be established. We conjecture it to be quasi-linear in the size of  $\gamma$  and  $\phi$ , provided the available features are fixed a priori.

Fourth, implementation techniques for  $FT$  at the level of the Warren abstract machine [2] need to be developed.

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