A THEORY OF COMPLETE LOGIC PROGRAMS WITH EQUALITY

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ABSTRACT

Incorporating equality into the unification process has added great power to automated theorem provers. We see a similar trend in logic programming where a number of languages are proposed with specialized or extended unification algorithms. There is a need to give a logical basis to these languages. We present here a general framework for logic programming with definite clauses, equality theories and generalized unification. The classic results for definite clause logic programs are extended in a simple and natural manner. The extension of the soundness and completeness of the negation-as-failure rule for complete logic programs is conceptually more delicate and represents the main result of this paper.

1 INTRODUCTION

In this paper, we consider generalized unification (e.g. Siekmann and Szabó 1982), i.e. unification of terms in equality theories, in the framework of logic programming. The theoretical foundation of incorporating equality into the unification process of theorem-proving was given by (Folkin 1972). The major result here was that for a set of clauses augmented with an equational theory, one can work on the clauses alone and yet have a complete inference system in a theorem-prover using a generalized unification algorithm, which respects the equational theory in question, and the usual resolution and paramodulation inference rules. As argued in both the above mentioned papers and (ICOT 1984), the study of generalized unification can have a tremendous practical significance. Our aims here are to present the counterpart of such results for logic programming, in particular for complete logic programs.

It is well-known that by restricting the logic

to definite clauses, we can have an elegant semantics for the resulting programming language (Van Emde and Kowalski 1976, Apt and van Emde 1982; Lassez and Maher 1984); furthermore, indications are that appropriate logic programming systems can be practically efficient (e.g. the various PROLOGs). In this paper, we show that the main desirable results for logic programming continue to hold in a more general framework of equality formulas in logic programs and generalized unification in the inference system. Thus this paper provides theoretical foundations for works such as those of (Kornfeld 1983) on equality in logic programs, and is relevant to works on functional programming in logic programming such as (Kahn 1981, Subrahmanyan and You 1984). Furthermore, the work of (Hansson and Haridi 1981) and (van Emde and Lloyd 1984) on various soundness results falls within this general framework which can be used to address the issues of completeness and performance as failure for Prolog II (Jaffar et al 1984).

The main result however concerns complete logic programs. A promising approach toward handling the assertion of negative facts in logic programming is in using the concept of complete logic programs (Clark 1978; Apt and van Emde 1982, Jaffar et al 1983). A particular attraction in the present efforts is that results on complete logic programs are associated with implementations of standard logic programs. That is, we gain additional expressive power for no additional cost.

One interesting (and indeed crucial as far as the present results are concerned) aspect of complete logic programs is the equality axioms embedded. In (Clark 1978) and (Jaffar et al 1983) these axioms enforce what is essentially, but not only, the Herbrand interpretations. These axioms are intimately connected with the standard unification algorithm which corresponds to syntactic equality. Upon close inspection one sees that these axioms are used to state explicitly those properties which are already built into the unification

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Here we consider complete logic programs incorporating equality axioms of the form of definite clauses. We show that a logic programming system (for the corresponding standard program) using appropriate generalized unification is a sound and complete inference mechanism for the complete program. In particular, the negation-as-failure rule in this general framework remains sound and complete.

The paper is organized as follows. In the next section we generalize the theory of definite clause logic programs whose equality theory is based on syntactic identity to easier programs with a class of equality theories. In section 3 we develop a theory of complete logic programs and unification-complete equality theories. This contrasts with present results which are restrictive in that complete programs are defined incorporating equality axioms which enforce interpretations which are essentially Herbrand ones.

2 LOGIC PROGRAMMING WITH EQUALITY AND GENERALIZED UNIFICATION

We use the symbols $V$, $\Sigma$ and $\Pi$ to denote the sets of variables, function symbols and non-logical predicate symbols respectively. Thus the latter set does not contain the symbol $\equiv$. $\tau(\Sigma)$ and $\tau(\Sigma \cup V)$ denote respectively the ground terms and the terms possibly containing variables. Throughout this paper we follow [Shoenfield 1987] for our mathematical logic terminology.

A definite clause logic program is defined in the usual manner, i.e. a finite set of definite clauses (van Emde and Kowalski 1976). Note that there are no $\equiv$ symbols in definite clauses. In this paper, we also consider definite clause equality theories. As usual, equations are of the form $s = t$ where $s$ and $t$ are terms over $\tau(\Sigma \cup V)$. A definite equality clause is of the form

\[ e \leftarrow e_1, e_2, \ldots, e_m \]

where $m \geq 0$ and all the atoms therein are equations. As usual, variables in definite equality clauses are implicitly universally quantified. We define a definite clause equality theory to be a possibly infinite set of definite equality clauses. See (Selman 1972) for some properties of definite clause equality theories. Finally, we define a logic program to be a pair $(P, E)$ where $P$ is a definite clause logic program and $E$ is a definite clause equality theory.

Generalized unification is defined here with respect to a definite clause equality theory $E$. Our definitions below are compatible with those in the literature (e.g. Huet and Oppen 1980, Siekmann and Szabo 1982). (However, such works usually consider only equational theories, i.e. universal closures of equations.) A substitution is defined to be a mapping from the set of variables $V$ into the set of terms $\tau(\Sigma \cup V)$. In what follows we sometimes (a) use obvious generalizations of substitutions to maps from terms into terms, and (b) speak of substitutions as equations, e.g. the substitution $\{x/y, y/u\}$ can be regarded as the set of equations $x = t$, $y = u$. An $E$-unifier for two terms $s$ and $t$ is a substitution $\theta$ such that $E \models s\theta = t\theta$.

We consider next the semantics of logic programs. As mentioned earlier, definite clause programs have an elegant formal semantics. The major reason for this is the existence of a canonical class of interpretations, namely the Herbrand ones, for the clauses. This follows from

\[ P \models p \iff \forall \tau(\Sigma) P \]

where $P$ is a definite clause logic program, $p$ an atom and where $\models_D$ denotes logical implication in the context of a fixed domain and functional assignment, in this case, $D$ is the Herbrand universe and functional assignment. This means that logical inference and refutations can be obtained within the purely syntactic framework of Herbrand interpretations. Furthermore, the existence of a least model for definite clauses provides a rigorous and simple declarative semantics for the corresponding programs (van Emde and Kowalski 1976).

Consider now our logic programs $(P, E)$ which contain equality theories. Here also we have a canonical class of interpretations. Let $\tau(\Sigma)/R$ denote the quotient of $\tau(\Sigma)$ by the congruence relation $R$. Thus the functional assignment is given by $f([t_1], \ldots, [t_m]) = [f(t_1, \ldots, t_m)]$ for all $m$-ary $f$ in $S$. It is well-known (see e.g. Loveland 1978) that

\[ (P, E) \models p \iff (P, E) \models_{\tau(\Sigma)/R} p \quad \text{for all } R \]

where $p$ is an atom, possibly an equation. What we require however, is a fixed domain and functional assignment, that is, a canonical congruence.
relation $R$ for a given program $(P, E)$. Clearly the only relations $R$ we need consider are given by the models of $E$. However, there is in general more than one model of $E$. The problem then, is to select a model which is representative of this collection. We prove in the lemma below that a least such $R$ exists. This then gives us Theorem 1, i.e. $R$ provides the desired canonical class of interpretations. This is in perfect analogy with canonicity of least models of definite clause programs.

**Lemma 1.** There exists a finest $\Sigma$-congruence over $\pi(\Sigma)$ generated by each definite clause equality theory $E$.

**Proof.** Consider models of $E$ over $\pi(\Sigma)$, and for our purposes here, a model is a set of pairs. Suppose now that $I$ is the intersection of a set of models of $E$. If $I$ is not a model itself, then some ground instance of a clause in $E$, say $e \leftarrow e_1, \ldots, e_n$, is falsified by $I$. This means that $e$ is not in $I$ and $e_1, \ldots, e_n$ are in $I$, contradicting the fact that $e$ is in the models of the set in question. The finest $\Sigma$-congruence then is given by the intersection of all models of $E$, with the obvious functional assignment such that $I([t_1], \ldots, [t_n]) = ([t_1], \ldots, [t_n])$ for all $n$-ary $f$ in $\Sigma$.

Although we consider only definite clause equality theories $E$ in this section, all the results below continue to hold for any open equality theory which has a finest $\Sigma$-congruence.

Let $R_0$ be the finest $\Sigma$-congruence generated by $E$.

**Theorem 1.** $(P, E) \models p$ iff $(P, E) \models_{\pi(\Sigma)/R_0} p$.

**Proof.** From the above remarks, it suffices to prove that $(P, E) \models_{\pi(\Sigma)/R_0} p$ iff $(P, E) \models_{\pi(\Sigma)/R} p$ for all $R$. The if part is trivial; consider the other part. For some $R$, let $I$ be any model over $\pi(\Sigma)/R$ for $(P, E)$ such that $p$ is false. Construct the following model $J$ over $\pi(\Sigma)/R_0$ by defining that $q([t])_{R_0}$ is true in $J$ iff $q([t])_{R}$ true in $I$ for all $n$-ary predicates symbol $q$. This is well-defined because $R_0$ is finer than $R$. That $J$ is indeed a model, in which $p$ is false, is now easy to see.

We now have the justification of working in a fixed domain. That is to say, we have that there is a canonical domain corresponding to a logic program, namely $\pi(\Sigma)/R_0$ where $R_0$ is the finest $\Sigma$-congruence over $\pi(\Sigma)$ generated by $E$. We henceforth may write $\pi(\Sigma)/E$ for $\pi(\Sigma)/R_0$ for any open equality theory $E$ which has a finest $\Sigma$-congruence $R_0$.

Let $\iota$ denote a sequence of terms $t_1, t_2, \ldots, t_n$, $n \geq 0$. We now give definitions with respect to a given logic program $(P, E)$. The $E$-base is $\bigcup_{p \in \pi(\Sigma)} \{ d \mid (\pi(\Sigma)/E)(d) \}$ over all $n$-ary predicate symbols $p$. An $E$-interpretation $I$ is a subset of the $E$-base. We write $[s]$ to denote the element in $\pi(\Sigma)/E$ assigned to the ground term $s$. Similarly, $[\iota]$ is a element of $(\pi(\Sigma)/E)^{\iota}$ and $[p][\iota]$ is an element of the $E$-base. Where $S$ is a set of ground terms, $[S]$ denotes $\{[s] \mid s \in S\}$.

We now define the appropriate generalizations of derivation sequences success and finite failure sets for our logic programs. We point out here that while these sets are defined in an operational manner, we do not address in this paper the issue of corresponding computational methods implementing them. In what follows, we write $t = u$ to mean $t_1 = u_1 \land t_2 = u_2 \land \ldots \land t_n = u_n$.

Thus we say $\iota$ $E$-unifies with $u$ to mean that $E \models (t_1 = u_1 \land \ldots \land t_n = u_n)$.

We observe here that an immediate consequence of Theorem 1 is that $\iota$ $E$-unifies with $\iota$ iff $[\iota] = [\iota]$.

For notational convenience, we assume that the variables in $V$ are not subscripted. A $(P, E)$-derivation sequence is a (finite or infinite) sequence of triples $<G_i, C_i, \theta_i>$, $i = 0, 1, \ldots$ such that (a) $G_i$ is of the form $B_1, \ldots, B_m$ where $m \geq 0$ and each $B_i$ is an atom, for all $1 \leq j \leq m$, (b) $C_i$ is a list of in clauses

$$A^{(0)} \leftarrow D^{(1)}, \ldots, D^{(n)}$$

$$A^{(2)} \leftarrow D^{(n)}, A^{(m)}$$

$$\ldots$$

$$A^{(m)} \leftarrow D^{(m)}, A^{(m)}$$

where each clause above is a clause from $P$ with variables renamed in that they are now subscripted with numbers never before used in subscripting in any $G_i$ where $j < i$, (c) $\theta_i$ is an $E$-unifier of $(B_1, \ldots, B_m)$ and $(A^{(1)}, \ldots, A^{(m)})$, and (d) $C_{i+1}$ is

$$(D^{(1)}, \ldots, D^{(m)}, D^{(1)}, \ldots, D^{(m)}, \ldots, D^{(m)})\theta_i$$

A derivation sequence is finitely failed with length $i$ if $\theta_i$ cannot be formed, that is, $B_1, \ldots, B_m$ and $(A^{(1)}, \ldots, A^{(m)})$ do not $E$-unify. A derivation sequence is successful if some $C_i$ is empty (i.e.
\( m = 0 \). Note that a derivation sequence is either successful, finitely failed or infinite.

The following defines the success, finite failure and general failure sets, denoted SS, FF and GF respectively, for a given logic program \((P, E)\).

\[ SS(P, E) = \{ p(s) : s \text{ is ground and there exists a successful } (P, E)\text{-derivation sequence of } p(s) \} \]

\[ FF(P, E) = \{ p(s) : s \text{ is ground and there exists a number } n \text{ such that all } (P, E)\text{-derivation sequences of } p(s) \text{ are finitely failed with length } \leq n \} \]

\[ GF(P, E) = \{ p(s) : s \text{ is ground and all } (P, E)\text{-derivation sequences of } p(s) \text{ are finitely failed} \} \]

Thus these definitions relate closely to resolution-like implementations of logic programming systems. It is necessary for us to consider the set GF because in general there is a ground atom which may not have an infinite derivation sequence and yet there is no number \( n \) such that all derivation sequences of this atom are finitely failed with length \( \leq n \). This possibility can arise because \( E \) can be such that there is an infinite set of maximally general \( E \)-unifiers for some pair of terms \( s \) and \( t \). However, if \( E \) is such that for all pairs of terms \( s \) and \( t \), there is a finite set of maximally general unifiers which subsume all the \( E \)-unifiers of \( s \) and \( t \), then GF is identical to FF.

In the standard framework, the success and finite failure sets have also been defined inductively. We give the appropriate generalizations here.

\[ SS_0(P, E) = \{ \} \]

\[ SS_{i+1}(P, E) = \{ p(t) : t \text{ is ground and there is a ground instance of a clause in } P \]

\[ p(u) \leftarrow B_1, \ldots, B_m \]

\[ \text{such that } t \text{ } E \text{-unifies with } u, \text{ and } \]

\[ B_k \in SS_i(P, E) \text{ for all } 1 \leq k \leq m \] \]

\[ SS(P, E) = \bigcup_{i=0}^{\infty} SS_i(P, E) \]

\[ FF_0(P, E) = \{ \} \]

\[ FF_{i+1}(P, E) = \{ p(t) : t \text{ is ground, and} \]

\[ \text{for each ground instance of a clause in } P \]

\[ p(u) \leftarrow B_1, \ldots, B_m \]

\[ \text{either } t \text{ does not } E \text{-unify with } u, \text{ or} \]

\[ B_k \in FF_i(P, E) \text{ for some } 1 \leq k \leq m \] \]

\[ FF(P, E) = \bigcup_{i=0}^{\infty} FF_i(P, E) \]

The proof of the following proposition is long but follows lines similar to the standard case and is therefore omitted.

**Proposition 1.**

(a) The two definitions of SS\((P, E)\) define the same set.

(b) The two definitions of FF\((P, E)\) define the same set.

One therefore might suspect that a corresponding (transfinite) inductive definition can be made for the set GF\((P, E)\). If \( \alpha \) and \( \beta \) denote not necessarily finite ordinals, then one could try:

\[ GF_0(P, E) = \{ \} \]

\[ GF_{\alpha+1}(P, E) = \]

\[ \text{if } (\alpha \neq 0 \text{ and } \alpha \text{ is not a limit ordinal}) \text{ then} \]

\[ \{ p(t) : t \text{ is ground, and} \]

\[ \text{for each ground instance of a clause in } P \]

\[ p(u) \leftarrow B_1, \ldots, B_m \]

\[ \text{either } t \text{ does not } E \text{-unify with } u, \text{ or} \]

\[ B_k \in GF_\beta(P, E) \text{ for some } 1 \leq k \leq m \] \]

\[ \text{else} \]

\[ \bigcup_{\beta < \infty} GF_\beta(P, E) \]

GF\((P, E)\) is such that \( A \in GF(P, E) \) iff \( A \in GF_\alpha(P, E) \) for some ordinal \( \alpha \).

Unfortunately, one can show that this definition is not equivalent to the above. Thus while we may use either one of the definitions for SS and FF, we have only one definition of GF in this paper. The problem of finding an inductive definition for GF\((P, E)\) remains.

As in [van Emden and Kowalski 1976] we make use of a function \( T \) in which terms most of the fundamental results can be framed. In the definition below, \( E \) denotes any open equality theory which has a finest congruence. \( T(P, E) \) is a function from and into \( E \)-interpretations.
\[ T_{[P,E]}(l) = \{p(t) : \text{there is a ground instance of a clause in } P \]
\[ p(t) \leftarrow B_1, \ldots, B_m \]
\[ \text{such that } [s] = \bar{c} \text{ and } \]
\[ [B_k] \in I \text{ for } 1 \leq k \leq m \} \]

We are now in a position to extend the classic results of standard logic programming theory. Since we have a canonical domain (cf. Theorem 1) for a given program \((P, E)\), the proofs of the lemmas leading to Theorem 2 and the theorem itself are simple extensions of their counterparts in the standard theory. Theorems 3 and 4 follow from the various lemmas and Proposition 1. The main new concept to be found in the lemmas below is generalized unification. Below we write \(I\) for some \(E\)-interpretation of \((P, E)\), and for brevity, we sometimes write \(T, SS, \text{ and } FF \) for \(T_{[P,E]} SS(P,E)\) and \(FF(P,E)\) respectively.

**Lemma 2.** \(T_{(P,E)}\) is continuous.

**Lemma 3.** \(I\) models \((P, E)\) iff \(T_{[P,E]}(l) \subseteq I\).

**Lemma 4.** For all \(i \geq 0\),
\[(a) p(t) \in SS_i \iff [p(t)] \in [SS_i].\]
\[(b) p(t) \in FF_i \iff [p(t)] \in [FF_i].\]

We write \(T_{(P,E)}(\omega) = \bigcup_{i=0}^{\omega} T_i(P,E)\), and \(T_i(\omega) = \bigcap_{i=0}^{\omega} T_i(E\text{-base})\). Using appropriate fixpoint theorems (see e.g. Levesque et al 1982), we have, from Lemma 2, that \(T_i(\omega)\) is the least fixpoint of \(T_i\) and, from Lemma 3, that \(T_i(\omega)\) is the least model of \((P, E)\), similarly to the standard case (van Emden and Kowalski 1976). Thus

**Theorem 2.** The least model of \((P, E)\) is equal to the least fixpoint of \(T_{(P,E)}\).

One more characterization of \(T_i(\omega)\) is given by

**Lemma 5.** \(T_{[P,E]}(\omega) = [SS(P,E)]\).

**Proof.** Let \(T_i\) denote \(T_i(P,E)\). We show \([SS_i] = T_i(\omega) = T_i(I)\) for all \(i \geq 0\) by induction; the lemma then follows. The base case \(i = 0\) is trivially proved. Now
\[ [SS_{i+1}] = \{p(t) : p(t) \in SS_{i+1}\}, \]
\[ = \{p(t) : \text{there is a ground instance of a clause in } P \]
\[ p(u) \leftarrow B_1, \ldots, B_m \]
\[ \text{such that } t \text{-E-unifies with } u, \text{ and } \]
\[ B_k \subseteq SS_i \text{ for all } 1 \leq k \leq m \}

by the definition of \(SS_{i+1}\)
\[ = \{p(t) : \text{there is a ground instance of a clause in } P \]
\[ p(u) \leftarrow B_1, \ldots, B_m \]
\[ \text{such that } [t] = [u], \text{ and } \]
\[ [B_k] \subseteq SS_i \text{ for all } 1 \leq k \leq m \}, \]
\[ \text{by Theorem 1 and Lemma 4(a) \]
\[ \subseteq T_i(\omega), \text{ by the induction hypothesis \]
\[ = T_{(P,E)}(\omega) \]

The following theorem establishes the soundness and completeness of a proof strategy based on \((P, E)\)-derivation sequences.

**Theorem 3.** If \(p(t)\) is a ground atom
\[ (P, E) \models p(t) \iff p(t) \in SS(P,E) \]
Finally we have a dual result for finite failure.

**Theorem 4.** If \(p(t)\) is a ground atom
\[ p(t) \in FF(P,E) \iff [p(t)] \in T_{[P,E]}(\omega). \]

3 COMPLETE LOGIC PROGRAMS WITH EQUALITY THEORIES

As mentioned before, generalized unification is usually defined over an equational theory \(E\), i.e. a set of open equations in some fixed alphabet \(\Sigma\). Two terms are then said to be \(E\)-unifiable if there is a ground substitution over \(\Sigma\) of the terms such that the ground instances are both in the same class of the finest \(\Sigma\)-congruence over \(\Sigma\) generated by \(E\). This does not however mean that if two terms are equal in another \(E\)-algebra modeling \(E\) then they are \(E\)-unifiable.

In this section we want to establish a relationship between failure and failure of atoms. We thus require in this section that an equality theory dictates that equality holds only if \(E\)-unification is possible. Formally, we say that an open equality theory \(E\) is **unification complete** over \(\Sigma\) if for every equality formula \(e\) of the form
\[ \exists y(s = t), \]
where \(y\) are the variables appearing in the terms \(s\) and \(t\), either \(E \models e\), or else there exists a non-empty and possibly infinite set \(\{\theta_i\}\) of \(E\)-unifiers of \(s\) and \(t\) such that
\[ \forall y((s = t) \rightarrow v(\theta_i)). \]
Note that the above expression means that in any model for \( E \), the following holds: If a valuation of the variables in terms of \( s \) and \( t \) is such that \( s = t \) in the model, then at least one of the \( E \)-unifiers \( \theta_q \) (looked upon as a set of equations) is also true in the model and valuation.

An augmented definite clause logic program consists of a conjunction of predicate definitions, exactly one for each predicate symbol in \( \Pi \). These definitions take one of the two forms:

\[
p_i(x) \leftarrow D_i \tag{1}
\]

or

\[
\neg p_i(x) \tag{2}
\]

where the \( \bar{x} \) are a list of \( n_i \) distinct variables, the \( p_i \) are \( n_i \)-ary predicate symbols, and the \( D_i \) are the definition bodies of \( p_i \). These bodies are each a disjunction of formulas of the form

\[
\exists y (x = \bar{t} \land B_1 \land B_2 \land \ldots \land B_m)
\]

where the \( B_j \) are atoms and \( y \) are the variables distinct from \( \bar{x} \) appearing in the formula. Note that these augmented programs are the same as the complete programs of (Clark 1978) except that we do not include his equality axioms. Finally, we can define our complete logic programs: these are of the form \((E^*, \Sigma)\) where \( E^* \) is an augmented definite clause program and \( \Sigma \) a unification complete equality theory.

It is well-known (see e.g. Clark 1978) how one obtains from a definite clause logic program a corresponding augmented version. The converse is also easy to define, that is to say, we can obtain from a given \( P^* \) an un augmented program \( P \). This is done as follows: For each predicate definition of type (2) in \( P^* \), obtain \( k \) definite clauses where \( k \) is the number of disjunctions in the definition. Then

\[
\exists y (x = \bar{t} \land B_1 \land B_2 \land \ldots \land B_m) \tag{3}
\]

is one such disjunct, obtain the corresponding definite clause

\[
p_i(\bar{x}) \leftarrow B_1 \ldots B_m \tag{4}
\]

Note that we do not construct any definite clauses from predicate definitions of type (2) in \( P^* \).

For unification complete equality theories \( \Sigma \), however, we have the following as the un-complete counterpart: \( E \equiv \{ e : e \text{ is a ground equation and } E^* \vdash e \} \). In the other direction, one can deal separately with each definite clause equality theory \( E \). For example, (Clark's 1978) axioms form a unification complete extension of the trivial equality theory consisting only of the usual equality axioms. In general however, there is no unique \( E^* \) corresponding to an \( E \). In what follows, we are only concerned with the \( E \) corresponding to some \( E^* \).

Since \( E^* \) is unification complete, we say that \( E^* \) is an \( E^* \)-interpretation to mean, as in section 2, that \( E \) has the domain given by \( r(\Sigma)/E^* \), this being the unique \( \Sigma \)-congruence over \( r(\Sigma) \) generated by \( E^* \). Thus \( I \) may be regarded as an interpretation of arbitrary formulas in the obvious way, i.e. \( \Sigma \) defines the domain and functional assignment by virtue of it being an \( E^* \)-interpretation, and \( I \) defines truth values via its elements. For brevity, we now write, when convenient, \( T, SS \) and \( GF \) for \( T(P, E), SS(P, E) \) and \( GF(P, E) \) respectively, where \((P, E)\) is the corresponding logic program to the complete logic program \((P^*, E^*)\) in question. We write \( p(s) \) to denote some ground atom. The following lemma generalizes a result of (Apt and van Emden 1982).

Lemma 6. If \( I \) is an \( E \)-interpretation, \( I \) is a fixpoint of \( T[p, E] \) if \( I \) is a model for \((P^*, E)\).

Proof. Let \( p \) be any non-logical \( n \)-ary predicate symbol in \( P^* \) and recall that there is only one definition of \( p \) there. If it is of the form (1), i.e.

\[
p(\bar{x}) \leftarrow \exists y \ C_i \text{ where each conjunction } C_i, 1 \leq i \leq k, \text{ is of the form (2), then this definition is satisfied by } I \text{ if and only if for all } d \in (r(\Sigma)/E^*)^n,
\]

\[
p(d) \in I \text{ for some } C_i \text{ and ground substitution } \theta, \theta[d] = [\theta] \text{ and } [\theta[b]] \in I \text{ for all } 1 \leq j \leq m.
\]

Since for each \( C_i \) there is a definite clause about \( p \) in \( P \) and vice versa, this is the same as

\[
p(d) \in I \iff p(\bar{d}) \in T[I] \text{ for all } \bar{d}.
\]

If however the definition of \( p \) is of the form (2), \( \neg p(\bar{x}) \) is satisfied by \( I \) if and only if for all \( \bar{d} \). By the definition of \( T[p, E] \), we have for each such \( p \) that for all \( E \)-interpretations \( J \) and all \( \bar{d} \), \( p(d) \notin T[I] \). Hence \((P^*, E)\) is satisfied by \( I \) if and only if \( T[p, E](I) = T[I] \).
I. 1

We are ready for two main theorems. The first proves the soundness and completeness of successful \((P_*, E_*)\)-derivations for positive atoms valid in \((P_*, E_*)\). We phrase our theorem thus:

**Theorem 5.** \((P_*, E_*) \models p(s) \iff p(s) \in SS\).

**Proof.** \((\rightarrow)\).

By the lemmas in section 2, if \(p(s) \notin SS\), then \(p(s) \notin I\) where \(I\) is the least \(E\)-model of \(P\). Again by these lemmas, \(I = T_{\mu_1}\) is a fixpoint of \(T\). Since \(I\) is also an \(E_1\)-interpretation, \(T_{(P, E_1)} = T\), and thus \(I\) is a fixpoint of \(T_{(P, E_*)}\). By lemma 0, \(I\) is a model for \((P_*, E_*)\).

\((\leftarrow)\).

It suffices to show that \(P_* \models P\). This is easily done along the following chain of reasoning: Any definition of the form \(p(x) \iff D\) in \(P_*\) contains the subformula

\[ p(x) \iff D \]

where \(D\) is of the form \(C_1 \lor \ldots \lor C_k\) for some \(k > 0\). This in turn is equivalent to the conjunction of

\[ p(x) \iff C_i \]

for \(1 \leq i \leq k\). That is to say we have a conjunction of formulas of the form

\[ p(x) \iff \exists y (x = i \land B_1 \land \ldots \land B_m). \]

Each such formula is equivalent to

\[ p(x) \iff \exists \bar{x} (i \land B_1 \land \ldots \land B_m). \]

by a suitable manipulation of quantifiers. Finally, this clearly implies the definite clause which appears in \(P\)

\[ p(i) \iff B_1, \ldots, B_m. \]

Since every definite clause in \(P\) is implied by some definition such as the \(p(x) \iff D\) above, we are done.

We now prove the soundness and completeness of generally failed \((P_*, E_*)\)-derivations for negative atoms valid in \((P_*, E_*)\). Thus we justify a form of the negation-as-failure rule (Clark 1978).

**Theorem 6.** \((P_*, E_*) \models \neg p(s) \iff p(s) \in GF\).

**Proof.** \((\rightarrow)\).

We prove that if for some model \(M\) of \((P_*, E_*)\), \(\exists \bar{y} (A_1 \land A_2 \land \ldots \land A_n)\) is true, then the goal \(A_1, A_2, \ldots, A_n\) has an infinite \((P_*, E_*)\)-derivation sequence or a successful one. It suffices to show that either \(A_1, \ldots, A_n\) is empty or we can have a derivation step starting from this goal and obtaining a goal \(B_1, B_2, \ldots, B_m\) such that \(\exists \bar{y} (B_1 \land B_2 \land \ldots \land B_m)\) is also true in \(M\). Repeated application of this construction proves the existence of a \((P_*, E_*)\)-derivation sequence which is either infinite or successful.

Suppose that for each \(1 \leq i \leq n\), \(A_i\) is of the form \(p^0(\ldots x^0(\ldots)\ldots)\) where \(x^0(\ldots)\) stands for a list of terms over \(\pi (\Sigma \cup V)\) whose variables appear in the list \(x^0(\ldots)\). Consider the definition in \(P_*\) of each of these (not necessarily distinct) predicate symbols \(p^0\): 

\[ p^0(\ldots x^0(\ldots)\ldots) \iff D \]

where \(D\) is a disjunction of formulas of the form

\[ \exists \bar{y} (x = i \land B_1 \land \ldots \land B_m). \]

Let \(V_0\) denote a valuation of \(A_1, \ldots, A_n\), i.e. an assignment of an element in the domain of \(M\) to each variable in \(x\) such that this conjunction is true in \(M\). Therefore, for each \(1 \leq i \leq n\), \(p^0(\ldots x^0(\ldots)\ldots)\) is true in \(M\) under this valuation \(V_0\). It thus follows that one of the formulas \(#(i)\) is true under this valuation. Hence the conjunction of these formulas

\[ \#(1) \land \\#(2) \land \ldots \land \\#(n) \]

is true in \(M\) under the valuation \(V_0\). Thus so is

\[ \exists \bar{y} (i \land B_1 \land \ldots \land B_m) \]

Hence the following is true in \(M\):

\[ \forall \bar{x} (\exists \bar{y} (i \land B_1 \land \ldots \land B_m) \iff (\ldots i \land B_1 \land \ldots \land B_m)). \]

where \(\bar{x}\) is the list of variables in \(\bar{x}(i) \ldots, \bar{x}(n)\). Our proof is now complete by three observations: (a) Since \(E_1\) is unification complete, there exists at least one \(E_1\)-unifier \(\theta\) for the equations
such that $\theta$ is true under any valuation for which
$V_{\theta}$ is a restriction. (b) Thus for each $1 \leq i \leq n$,
$(B_{1}^{(1)} \land B_{2}^{(1)} \land \ldots \land B_{n}^{(1)})$ is true in $M$ for some valuation
of the variables therein. (c) There exists in $P$ definite clauses of the form

\[ f^{(1)}(\theta^{(1)}) \leftarrow E_{1}^{(1)}, B_{2}^{(1)}, \ldots, B_{n}^{(1)} \]

for all $1 \leq i \leq n$. Putting (a), (b) and (c) together, we may conclude that from $A_{1}$, $\ldots$, $A_{n}$ and these definite clauses above, we can (P, E)-derive

$(B_{1}^{(1)}, \ldots, B_{n}^{(1)}, \ldots, B_{1}^{(n)}, \ldots, B_{n}^{(n)})$.

(\rightarrow).

Assuming that $p[s] \notin GF$, we now construct a model for (P+, E*) in which $p[s]$ is true. We may as well assume that $p[s] \notin SS$. By the results above, $p[s]$ is the first goal in an infinite derivation sequence $<G_{s}, C_{1}, \ell_{1}>$, $i = 0, 1, \ldots$. Recall that by our variable renaming convention, there are no common variables in $C_{1}$ and $C_{j}$ where $i \neq j$. Let $E_{i}$ denote a finite set of ground equations over a larger alphabet $\Sigma+$ in that $E_{i}$ is obtained from $\theta_{i}$ (locked upon as a set of equations) by replacing each distinct occurrence of a variable $x_{j}$ with a distinct new constant symbol $c_{j}$. In what follows we make use of the fundamental property of the $\theta_{i}$:

\[ E_{i} \models \exists(x_{1} \land c_{2} \land \ldots \land c_{n}) \]

for any finite $n$.

We now complete the proof in two main steps. Firstly, we show that $E^{*+} = E^{*} \cup \{E_{i}\}$ is consistent. Thus since $E^{*+}$ is open, we may have $(E^{*+})$-interpretations $I_{0}$. Secondly, we build a fixpoint $I$ of $T_{E^{*+}}$. Using lemma 7, we are done.

To show that $E^{*+}$ is consistent, it suffices to show, by the Compactness Theorem, that $E^{*+} = E^{*} \cup \{E_{1}^{*}, \ldots, E_{n}^{*}\}$ is consistent for all finite $n$. Let $A$ be any closed formula over $\pi(\Sigma)$. Consider the following chain of reasoning:

\[ E^{*+} \models A \]
\[ A^{(1)} \leftarrow D^{(1)}_1, \ldots, D^{(1)}_{\alpha_1} \]
\[ A^{(2)} \leftarrow D^{(2)}_1, \ldots, D^{(2)}_{\alpha_2} \]
\[ \vdots \]
\[ A^{(m)} \leftarrow D^{(m)}_1, \ldots, D^{(m)}_{\alpha_m} \]

Recall that we rename variables so that they are subscripted. Since the derivation sequence is infinite, \( G_{i+1} \) exists and must be of the form

\[ (D^{(1)}_1, \ldots, D^{(1)}_{\alpha_1}, \ldots, D^{(m)}_1, \ldots, D^{(m)}_{\alpha_m}) \]

where \( \theta_i \) is an E-unifier of \( G_i \) and \( A^{(1)}, \ldots, A^{(m)} \).

Since \( [G_{i+1}] \subseteq I_0 \), we have \([A^{(1)}\theta_i], \ldots, A^{(m)}\theta_i] \subseteq T(I_0)\).

It remains to prove that \([A^{(j)}\theta_i] = [B_i] \) for all \( 1 \leq j \leq m \). Suppose now that \( \theta_i \) is of the form

\[ \{x_1/t_1(x), x_2/t_2(x), \ldots, x_n/t_n(x)\} \]

where \( x \) is the list of all subscripted variables appearing here. By construction, \( E \) is of the form

\[ c_1 = t_1(c), \ldots, c_n = t_n(c) \]

where \( c \) is the list of new constants corresponding to the \( x \). Since \( E^+ \) contains \( E \), it follows that for all \( 1 \leq j \leq m \),

\[ [B_i] = [B_i\theta_i] \]

Since also \( \theta_i \) E-unifies \( A^{(j)} \) and \( B_j \), we obtain \([A^{(j)}\theta_i] = [B_i] \) for all \( 1 \leq j \leq m \). Thus \([E_1, \ldots, E_m] \subseteq T(I_0)\) and hence \( I_0 \subseteq T(I_0)\).

Finally, we can use the Knaster-Tarski theorem about fixpoints for monotonic functions to show that there exists an \( I \) which contains \( I_0 \) such that \( T(I) = I \).

**REFERENCES**


